Interpreting Dreams of Abstract Machines

Bernard Sufrin, University of Oxford

December 2015

A programmer's reading of the 1843 Lovelace-Menabrea sketch of the Analytical Engine

It is desirable to guard against the possibility of exaggerated ideas that might arise

In considering any new subject, there is frequently a tendency, first, to overrate what we find to be already interesting or remarkable; and, secondly, by a sort of natural reaction, to undervalue the true state of the case, when we discover that our notions have surpassed those that were really tenable.

For the programmer, a patient reading will yield:

- A (blurred) snapshot of Babbage's *developing idea* of an Analytical Engine
- Several related notations for *programming* calculations for the Engine
- The derivation of an algorithm, and its *near-implementation* as a program (for the Analytical Engine)

Context

By the time of [the Greeks] several nontrivial algorithms had been studied rather thoroughly [...] The description of algorithms was always informal, however, rendered in natural language.

During the ensuing [post-Greek] centuries, mathematicians never did invent a good notation for dynamic processes [...]. When a procedure involved nontrivial sequences of decisions, the available methods for precise description remained informal and rather cumbersome.

Mathematicians would traditionally present the control mechanisms of algorithms informally, and the computation involved would be expressed by means of equations.
Knuth & Trabb-Pardo [1]

From: Allan Bromley
To: Maurice Wilkes
Date: 19th July 2000

```
I have just worked again through Babbage's notebooks [con-
cerning] his work on the Analytical Engines 1857-70.  I have
been greatly disappointed by what I have found ...

Through most of the period the operation cards provided for
only four operations:  addition, subtraction, multiplication,
and division.

...

To save operation cards, each could specify the number of
times it was to be repeated.  Small loops of operation cards
came late but were replaced by each operation card specifying
how far forward or backward to go for the next.

There is no suggestion [in the notebooks] of conditionals
available to the user.
```

Towards the Abstract Machine

Menabrea's view:

> *[T]he machine is not a thinking being, but simply an automaton ...*
>
> *This being fundamental, one of the earliest researches [Babbage] had to undertake, was that of finding means for effecting the division of one number by another without using the method of guessing indicated by the usual rules of arithmetic. The difficulties of effecting this combination were far from being among the least; but upon it depended the success of every other.*
>
> *Under the impossibility of my here explaining the process through which this end is attained, ...*
>
> *... we must limit ourselves to admitting that the first four operations of arithmetic, that is addition, subtraction, multiplication and division, can be performed in a direct manner through the intervention of the machine.*

Lovelace contrasts Menabrea's abstract exposition to Lardner's [2]

> *M. Menabrea, on the contrary, exclusively developes the analytical view;* ***taking it for granted that mechanism is able to perform certain processes, but without attempting to explain how****;*
> (Note A)

"You needn't dream of the clink and clank of the machinery to understand its function."

Lovelace's understanding of the relationship between mechanism and function:

> *It is obvious that, in the invention of a calculating engine, these two branches of the subject are equally essential fields of investigation, and that on their mutual adjustment, one to the other, must depend all success. ... They are indissolubly connected, though so different in their intrinsic nature, that perhaps the same mind might not be likely to prove equally profound or successful in both.*
> (Note A)

A concrete detail intrudes

Each instruction to the machine involves three columns of the store

> *In general, if we have a series of columns consisting of discs, which columns we will designate as* V_0, V_1, V_2, V_3, V_4, *&c., we may require, for instance, to divide the number written on the column* V_1 *by that on the column* V_4*, and to obtain the result on the column* V_7.

It takes two mechanisms to specify the instruction

> *To effect this operation, we must impart to the machine two distinct arrangements; through the first it is prepared for executing a division, and through the second the columns it is to operate on are indicated to it, and also the column on which the result is to be represented.*
> (Menabrea)

One specifies the operation; *the other* specifies the operand and result columns

The two mechanisms must be distinct

If this division is to be followed, for example, by the addition of two numbers taken on other columns, ***the two original arrangements of the machine must be simultaneously altered****.*

If, on the contrary, a series of operations of the same nature is to be gone through, then ***the first of the original arrangements will remain****,* ***and the second alone must be altered****.* [Menabrea]

Menabrea's first programmed Calculation

$$x = \frac{dn' - d'n}{n'm - nm'} \text{ into } V_{14}$$

$$V_0 = m,\ V_1 = \mathsf{n},\ V_2 = d,\ V_3 = m',\ V_4 = n',\ V_5 = d'$$

Op#	Op	Variables		Progress
1	$\times$	$V_2 \times V_4 =$	$V_8 \cdots$	$= dn'$
2	$\times$	$V_5 \times V_1 =$	$V_9 \cdots$	$= d'n$
3	$\times$	$V_4 \times V_0 =$	$V_{10} \cdots$	$= n'm$
4	$\times$	$V_1 \times V_3 =$	$V_{11} \cdots$	$= nm'$
5	$-$	$V_8 - V_9 =$	$V_{12} \cdots$	$= dn' - d'n$
6	$-$	$V_{10} - V_{11} =$	$V_{13} \cdots$	$= n'm - nm'$
7	$\div$	$V_{12} \div V_{13} =$	$V_{14} \cdots$	$= x = \frac{dn'-d'n}{n'm-nm'}$

"When two numbers have been thus written on two distinct columns, we may propose to combine them arithmetically with each other, and to obtain the result on a third column."

Menabrea's first programmed Calculation

$$x = \frac{dn' - d'n}{n'm - nm'} \quad \text{into} \quad V_{14}$$

$$V_0 = m,\ V_1 = \text{n},\ V_2 = d,\ V_3 = m',\ V_4 = n',\ V_5 = d'$$

Op#	Op	Variables		Progress
1	$\times$	$V_2 \times V_4 =$	$V_8 \cdots$	$= dn'$
2	$\times$	$V_5 \times V_1 =$	$V_9 \cdots$	$= d'n$
3	$\times$	$V_4 \times V_0 =$	$V_{10} \cdots$	$= n'm$
4	$\times$	$V_1 \times V_3 =$	$V_{11} \cdots$	$= nm'$
5	$-$	$V_8 - V_9 =$	$V_{12} \cdots$	$= dn' - d'n$
6	$-$	$V_{10} - V_{11} =$	$V_{13} \cdots$	$= n'm - nm'$
7	$\div$	$V_{12} \div V_{13} =$	$V_{14} \cdots$	$= x = \frac{dn'-d'n}{n'm-nm'}$

"takes 3 operation cards if desired"

Lovelace's account of the general-purpose nature of the engine

That portion of the Analytical Engine here alluded to is called the store-house. It contains an indefinite number of the columns of discs described by M. Menabrea. ... We may conveniently represent the columns of discs on paper in a diagram like the following –

V_1	V_2	V_3	V_4	$\cdots$	Column names – values will vary
$-$	$+$	$+$	$+$		Sign of the current value
0	0	0	0	$\cdots$	One digit of the current value
0	0	0	0		″
0	0	9	0		″
5	7	8	0	$\cdots$	″
a	n	x			Captions explaining which variables are allocated to which columns

... ready to receive at any moment, by means of cards constituting a portion of its mechanism ... the impress of whatever special function we may desire to develope or to tabulate.

To calculate ax^n with these variables on its columns

V_1	V_2	V_3	V_4
a	n	x	ax^n

... Six multiplications to get x^n; then one multiplication to get $a \cdot x^n$

$$(\times, \times, \times, \times, \times, \times, \times) \quad \textit{or} \quad 7(\times)$$

... For x^{an} the operations would be $34(\times)$

The multiplications would, however, at successive stages in the solution of the problem, operate on pairs of numbers, derived from different columns. In other words, the same operation would be performed on different subjects of operation.
(Note B)

Menabrea's complete table

Calculating $x = \frac{dn'-d'n}{n'm-nm'}$, $y = \frac{d'm-dm'}{n'm-nm'}$

Primitive Data	Op'n #	Operation Cards: Card #	Operation Cards: Op'n	Variable Cards: Acted on	Variable Cards: Result	Variable Cards: Changes	Statement of Results
${}^1V_0 = m$	1	1	$\times$	${}^1V_0 \times {}^1V_4 =$	${}^1V_6$	$\left\{ {}^1V_0 = {}^1V_0 \atop {}^1V_4 = {}^1V_4 \right\}$	${}^1V_0 = mn'$
${}^1V_1 = n$	2	″	$\times$	${}^1V_3 \times {}^1V_1 =$	${}^1V_7$	$\left\{ {}^1V_3 = {}^1V_3 \atop {}^1V_1 = {}^1V_1 \right\}$	${}^1V_7 = m'n$
${}^1V_2 = d$	3	″	$\times$	${}^1V_2 \times {}^1V_4 =$	${}^1V_8$	$\left\{ {}^1V_2 = {}^1V_2 \atop {}^1V_4 = {}^0V_4 \right\}$	${}^1V_8 = dn'$
${}^1V_3 = m'$	4	″	$\times$	${}^1V_5 \times {}^1V_1 =$	${}^1V_9$	$\left\{ {}^1V_5 = {}^1V_5 \atop {}^1V_1 = {}^0V_1 \right\}$	${}^1V_9 = d'n$
${}^1V_4 = n'$	5	″	$\times$	${}^1V_0 \times {}^1V_5 =$	${}^1V_{10}$	$\left\{ {}^1V_0 = {}^0V_0 \atop {}^1V_5 = {}^0V_5 \right\}$	${}^1V_{10} = d'm$
${}^1V_5 = d'$	6	″	$\times$	${}^1V_2 \times {}^1V_3 =$	${}^1V_{11}$	$\left\{ {}^1V_2 = {}^0V_2 \atop {}^1V_3 = {}^0V_3 \right\}$	${}^1V_{11} = dm'$
	7	2	$-$	${}^1V_6 - {}^1V_7 =$	${}^1V_{12}$	$\left\{ {}^1V_6 = {}^0V_6 \atop {}^1V_7 = {}^0V_7 \right\}$	${}^1V_{12} = mn' - m'n$
	8	″	$-$	${}^1V_8 - {}^1V_9 =$	${}^1V_{13}$	$\left\{ {}^1V_8 = {}^0V_8 \atop {}^1V_9 = {}^0V_9 \right\}$	${}^1V_{13} = dn' - d'n$
	9	″	$-$	${}^1V_{10} - {}^1V_{11} =$	${}^1V_{14}$	$\left\{ {}^1V_{10} = {}^0V_{10} \atop {}^1V_{11} = {}^0V_{11} \right\}$	${}^1V_{14} = d'm - dm'$
	10	3	$\div$	${}^1V_{13} \div {}^1V_{12} =$	${}^1V_{15}$	$\left\{ {}^1V_{13} = {}^0V_{13} \atop {}^1V_{12} = {}^1V_{12} \right\}$	${}^1V_{15} = \frac{dn'-d'n}{mn'-m'n} = x$
	11	″	$\div$	${}^1V_{14} \div {}^1V_{12} =$	${}^1V_{16}$	$\left\{ {}^1V_{14} = {}^0V_{14} \atop {}^1V_{12} = {}^0V_{12} \right\}$	${}^1V_{16} = \frac{d'm-dm'}{mn'-m'n} = y$

Simplification

Op'n #	Card #	Op'n	Acted on	Result	Statement of Results
	Operation Cards		Variable Cards		
1	1	$\times$	$V_0 \times V_4$	$\rightarrow V_6$	$V_0 = mn'$
2	"	$\times$	$V_3 \times V_1$	$\rightarrow V_7$	$V_7 = m'n$
3	"	$\times$	$V_2 \times V_4$	$\rightarrow V_8$	$V_8 = dn'$
4	"	$\times$	$V_5 \times V_1$	$\rightarrow V_9$	$V_9 = d'n$
5	"	$\times$	$V_0 \times V_5$	$\rightarrow V_{10}$	$V_9 = d'n$
6	"	$\times$	$V_2 \times V_3$	$\rightarrow V_{11}$	$V_{11} = dm'$
7	2	$-$	$V_6 - V_7$	$\rightarrow V_{12}$	$V_{12} = mn' - m'n$

◦ The appearance of V_i in an "acted on" variable signifies a read-and-clear

Lovelace's table for $x = \frac{dn'-d'n}{n'm-nm'}$, $y = \frac{d'm-dm'}{n'm-nm'}$

Number of Operations	Nature of Operations	Variables for Data						Working Variables									Variables for Results	
		$^1V_0$	$^1V_1$	$^1V_2$	$^1V_3$	$^1V_4$	$^1V_5$	$^0V_6$	$^0V_7$	$^0V_8$	$^0V_9$	$^0V_{10}$	$^0V_{11}$	$^0V_{12}$	$^0V_{13}$	$^0V_{14}$	$^0V_{15}$	$^0V_{16}$
		+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		m	n	d	m'	n'	d'										$\frac{dn'-d'n}{mn'-m'n} = x$	$\frac{d'm-dm}{mn'-m'n} = y$
1	$\times$	m	....	....	....	n'	....	mn'										
2	$\times$	....	n	....	m'	....	....	....	$m'n$									
3	$\times$	....	....	d	....	....	....	....	....	dn'								
4	$\times$	....	0	....	....	....	d'	....	....	....	$d'n$							
5	$\times$	0	....	....	....	....	0	....	....	....	....	$d'm$						
6	$\times$	....	....	0	0	....	....	....	....	....	....	.....	dm'					
7	$-$	....	....	....	....	....	....	0	0	....	....	.....	.....	$(mn'-m'n)$				
8	$-$	....	....	....	....	....	....	....	....	0	0	.....	.....	....	$(dn'-d'n)$			
9	$-$	....	....	....	....	....	....	....	....	....	....	0	0	....	....	$(d'm-dm')$		
10	$\div$	....	....	....	....	....	....	....	....	....	....	.....	.....	$(mn'-m'n)$	0	....	$\frac{dn'-d'n}{mn'-m'n} = x$	
11	$\div$	....	....	....	....	....	....	....	....	....	....	.....	.....	0	....	0	....	$\frac{d'm-dm'}{mn'-m'n} = y$

Essence of Lovelace's table for $x = \frac{dn'-d'n}{n'm-nm'}$, $y = \frac{d'm-dm'}{n'm-nm'}$

		Variables for Data						Working Variables									Variables for Results	
		m	n	d	m'	n'	d'										$\frac{dn'-d'n}{mn'-m'n} = x$	$\frac{d'm-dm}{mn'-m'n} = y$
1	$\times$	m	....	....	....	n'	....	mn'										
2	$\times$	....	n	....	m'	....	....	....	$m'n$									
3	$\times$	....	....	d	....	....	....	....	....	dn'								
4	$\times$	....	0	....	....	....	d'	....	....	....	$d'n$							
5	$\times$	0	....	....	....	....	0	....	....	....	....	$d'm$						
6	$\times$	....	....	0	0	....	....	....	....	....	....	.....	dm'					
7	$-$	....	....	....	....	....	....	0	0	....	....	.....	.....	$(mn' - m'n)$				
8	$-$	....	....	....	....	....	....	....	....	0	0	.....	.....	....	$(dn' - d'n)$			
9	$-$	....	....	....	....	....	....	....	....	....	....	0	0	....	....	$(d'm - dm')$		
10	$\div$	....	....	....	....	....	....	....	....	....	....	.....	.....	$(mn' - m'n)$	0	....	$\frac{dn'-d'n}{mn'-m'n} = x$	
11	$\div$	....	....	....	....	....	....	....	....	....	....	.....	.....	0	....	0	....	$\frac{d'm-dm'}{mn'-m'n} = y$

(Working and result variables allocated, read&clear differentiated from read&refresh by 0 appearing in the column)

Understanding the Bernoulli Number Table

(with Rainer Glaschick, Paderborn)

Number of Operation	Nature of Operation	Variables acted upon	Variables receiving results	Indication of change in the value on any Variable	Statement of Results	Data			Working Variables										Result Variables			
						$^1V_1$	$^1V_2$	$^1V_3$	$^0V_4$	$^0V_5$	$^0V_6$	$^0V_7$	$^0V_8$	$^0V_9$	$^0V_{10}$	$^0V_{11}$	$^0V_{12}$	$^0V_{13}\cdots$	$^1V_{21}$	$^1V_{22}$	$^1V_{23}$	$^0V_{24}\cdots$
						○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
						0	0	0	0	0	0	0	0	0	0	0	0	0	B in a dec. fract.	B in a dec. fract.	B in a dec. fract.	0
						0	0	0	0	0	0	0	0	0	0	0	0	0				0
						1	2	4	0	0	0	0	0	0	0	0	0	0				0
						1	2	n											B_1	B_3	B_5	B_7
1	×	$^1V_2 \times {}^1V_3$	$^1V_4, {}^1V_5, {}^1V_6$	$\left\{\begin{matrix} ^1V_2 = {}^1V_2 \\ ^1V_3 = {}^1V_3 \end{matrix}\right\}$	$= 2n$	......	2	n	$2n$	$2n$	$2n$											
2	−	$^1V_4 - {}^1V_1$	$^2V_4$	$\left\{\begin{matrix} ^1V_4 = {}^2V_4 \\ ^1V_1 = {}^1V_1 \end{matrix}\right\}$	$= 2n - 1$	1	......	......	$2n-1$													
3	+	$^1V_5 + {}^1V_1$	$^2V_5$	$\left\{\begin{matrix} ^1V_5 = {}^2V_5 \\ ^1V_1 = {}^1V_1 \end{matrix}\right\}$	$= 2n + 1$	1	......	......	......	$2n+1$												
4	÷	$^2V_5 \div {}^2V_4$	$^1V_{11}$	$\left\{\begin{matrix} ^2V_5 = {}^0V_5 \\ ^2V_4 = {}^0V_4 \end{matrix}\right\}$	$= \frac{2n-1}{2n+1}$	......	......	......	0	0	......	......	......	......	......	$\frac{2n-1}{2n+1}$						
5	÷	$^1V_{11} \div {}^1V_2$	$^2V_{11}$	$\left\{\begin{matrix} ^1V_{11} = {}^2V_{11} \\ ^1V_2 = {}^1V_2 \end{matrix}\right\}$	$= \frac{1}{2} \cdot \frac{2n-1}{2n+1}$	......	2	......	......	......	......	......	......	......	......	$\frac{1}{2} \cdot \frac{2n-1}{2n+1}$						
6	−	$^0V_{13} - {}^2V_{11}$	$^1V_{13}$	$\left\{\begin{matrix} ^2V_{11} = {}^0V_{11} \\ ^0V_{13} = {}^1V_{13} \end{matrix}\right\}$	$= -\frac{1}{2} \cdot \frac{2n-1}{2n+1} = A_0$	......	......	......	......	......	......	......	......	......	......	0	......	$= -\frac{1}{2} \cdot \frac{2n-1}{2n+1} = A_0$				
7	−	$^1V_3 - {}^1V_1$	$^1V_{10}$	$\left\{\begin{matrix} ^1V_3 = {}^1V_3 \\ ^1V_1 = {}^1V_1 \end{matrix}\right\}$	$= n - 1 (= 3)$	1	......	n	......	......	......	......	......	......	$n-1$							
8	+	$^1V_2 + {}^0V_7$	$^1V_7$	$\left\{\begin{matrix} ^1V_2 = {}^1V_2 \\ ^0V_7 = {}^1V_7 \end{matrix}\right\}$	$= 2 + 0 = 2$	......	2	......	......	......	......	2										
9	÷	$^1V_6 \div {}^1V_7$	$^3V_{11}$	$\left\{\begin{matrix} ^1V_6 = {}^1V_6 \\ ^0V_{11} = {}^3V_{11} \end{matrix}\right\}$	$= \frac{2n}{2} = A_1$	......	......	......	......	......	$2n$	2	......	......	......	$\frac{2n}{2} = A_1$						
10	×	$^1V_{21} \times {}^3V_{11}$	$^1V_{12}$	$\left\{\begin{matrix} ^1V_{21} = {}^1V_{21} \\ ^3V_{11} = {}^3V_{11} \end{matrix}\right\}$	$= B_1 \cdot \frac{2n}{2} = B_1A_1$	......	......	......	......	......	......	......	......	......	......	$\frac{2n}{2} = A_1$	$B_1 \cdot \frac{2n}{2} = B_1A_1$	......	B_1			
11	+	$^1V_{12} + {}^1V_{13}$	$^2V_{13}$	$\left\{\begin{matrix} ^1V_{12} = {}^0V_{12} \\ ^1V_{13} = {}^2V_{13} \end{matrix}\right\}$	$= -\frac{1}{2} \cdot \frac{2n-1}{2n+1} + B_1 \cdot \frac{2n}{2}$	......	......	......	......	......	......	......	......	......	......	......	0	$\left\{-\frac{1}{2} \cdot \frac{2n-1}{2n+1} + B_1 \cdot \frac{2n}{2}\right\}$				
12	−	$^1V_{10} - {}^1V_1$	$^2V_{10}$	$\left\{\begin{matrix} ^1V_{10} = {}^2V_{10} \\ ^1V_1 = {}^1V_1 \end{matrix}\right\}$	$= n - 2 (= 2)$	1	......	......	......	......	......	......	......	......	$n-2$							
13	−	$^1V_6 - {}^1V_1$	$^2V_6$	$\left\{\begin{matrix} ^1V_6 = {}^2V_6 \\ ^1V_1 = {}^1V_1 \end{matrix}\right\}$	$= 2n - 1$	1	......	......	......	......	$2n-1$											
14	+	$^1V_1 + {}^1V_7$	$^2V_7$	$\left\{\begin{matrix} ^1V_1 = {}^1V_1 \\ ^1V_7 = {}^2V_7 \end{matrix}\right\}$	$= 2 + 1 = 3$	1	......	......	......	......	......	3										
15	÷	$^2V_6 \div {}^2V_7$	$^1V_8$	$\left\{\begin{matrix} ^2V_6 = {}^2V_6 \\ ^2V_7 = {}^2V_7 \end{matrix}\right\}$	$= \frac{2n-1}{3}$	......	......	......	......	......	$2n-1$	3	$\frac{2n-1}{3}$									
16	×	$^1V_8 \times {}^3V_{11}$	$^4V_{11}$	$\left\{\begin{matrix} ^1V_8 = {}^0V_8 \\ ^3V_{11} = {}^4V_{11} \end{matrix}\right\}$	$= \frac{2n}{2} \cdot \frac{2n-1}{3}$	......	......	......	......	......	......	......	0	......	......	$\frac{2n}{2} \cdot \frac{2n-1}{3}$						
17	−	$^2V_6 - {}^1V_1$	$^3V_6$	$\left\{\begin{matrix} ^2V_6 = {}^3V_6 \\ ^1V_1 = {}^1V_1 \end{matrix}\right\}$	$= 2n - 2$	1	......	......	......	......	$2n-2$											
18	+	$^1V_1 + {}^2V_7$	$^3V_7$	$\left\{\begin{matrix} ^2V_7 = {}^3V_7 \\ ^1V_1 = {}^1V_1 \end{matrix}\right\}$	$= 3 + 1 = 4$	1	......	......	......	......	......	4										
19	÷	$^3V_6 \div {}^3V_7$	$^1V_9$	$\left\{\begin{matrix} ^3V_6 = {}^3V_6 \\ ^3V_7 = {}^3V_7 \end{matrix}\right\}$	$= \frac{2n-2}{4}$	......	......	......	......	......	$2n-2$	4	......	$\frac{2n-2}{4}$	......							
20	×	$^1V_9 \times {}^4V_{11}$	$^5V_{11}$	$\left\{\begin{matrix} ^1V_9 = {}^0V_9 \\ ^4V_{11} = {}^5V_{11} \end{matrix}\right\}$	$= \frac{2n}{2} \cdot \frac{2n-1}{3} \cdot \frac{2n-2}{4} = A_3$	......	......	......	......	......	......	......	......	0		$\left\{\frac{2n}{2} \cdot \frac{2n-1}{3} \cdot \frac{2n-2}{4}\right\} = A_3$						
21	×	$^1V_{22} \times {}^5V_{11}$	$^0V_{12}$	$\left\{\begin{matrix} ^1V_{22} = {}^1V_{22} \\ ^0V_{12} = {}^2V_{12} \end{matrix}\right\}$	$= B_3 \cdot \frac{2n}{2} \cdot \frac{2n-1}{3} \cdot \frac{2n-2}{4} = B_3A_3$	......	......	......	......	......	......	......	......	......	......	0	B_3A_3	......	......	B_3		
22	+	$^2V_{12} + {}^2V_{13}$	$^3V_{13}$	$\left\{\begin{matrix} ^2V_{12} = {}^0V_{12} \\ ^2V_{13} = {}^3V_{13} \end{matrix}\right\}$	$= A_0 + B_1A_1 + B_3A_3$	......	......	......	......	......	......	......	......	......	......	......	0	$\{A_0 + B_1A_1 + B_3A_3\}$				
23	−	$^2V_{10} - {}^1V_1$	$^3V_{10}$	$\left\{\begin{matrix} ^2V_{10} = {}^3V_{10} \\ ^1V_1 = {}^1V_1 \end{matrix}\right\}$	$= n - 3 (= 1)$	1	......	......	......	......	......	......	......	......	$n-3$							
					Here follows a repetition of Operations thirteen to twenty-three																	
24	+	$^4V_{13} + {}^0V_{24}$	$^1V_{24}$	$\left\{\begin{matrix} ^4V_{13} = {}^0V_{13} \\ ^0V_{24} = {}^1V_{24} \end{matrix}\right\}$	$= B_7$	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	B_7
25	+	$^1V_1 + {}^1V_3$	$^1V_3$	$\left\{\begin{matrix} ^1V_1 = {}^1V_1 \\ ^1V_3 = {}^1V_3 \\ ^5V_6 = {}^0V_6 \\ ^5V_7 = {}^0V_7 \end{matrix}\right.$	$= n + 1 = 4 + 1 = 5$ by a Variable-card. by a Variable-card.	1	......	$n+1$	......	......	0	0										

The Bottom Line

- Lovelace's commentary on her table explains the computation of B_7, from B_1, B_3, B_5.
- This style of explanation is in line with a tradition that goes back to the Babylonians
- Lovelace clearly intends to show that the table can be interpreted more generally – with lines 13-25 being the *essence of the body of a loop* that can calculate and store each (nonzero) Bernoulli number given that its predecessors have already been stored:
  - Despite finding *bugs in the published table* we have each shown, by building an emulation of a line-by-line transliteration of the (corrected) program, that this is the case.
  - Had the Engine existed, we believe that Lovelace would have noticed the bugs very quickly.
  - Two of the bugs are very serious – but we have been unable to find an acknowledgement of them in any of the literature on the Bernoulli "program".

- Our emulations take Lovelace's intended generality into account.

- But remember that in the Analytical Engine as described by Menabrea-Lovelace:
  - ▷ It is not possible to compute a number to use as the "*address*" of a column
  - ▷ The columns have "*names*" that just happen to be numbers

Bernoulli numbers & the Engine's 50-digit numeric precision

"If only Babbage had finished the implementation of conditionals and iteration **and** solved the problem of using computed numbers to denote columns ("indexing") perhaps things might have turned out differently for the Engine, for him, and for Lovelace."

$$B_{59} = \frac{-121523314048375557204030499407982024604 1491}{56786730}$$

$$B_{89} = \frac{1179057279021082799884123351249215083775254949669647116231545215727922535}{272118}$$

$$B_{181} = \frac{\begin{array}{c}42772692793491925411373044006286293483274\\68135828402291661683018622451659989595510\\71291581043623872113954696355865526038432\\89887732196880914435296265313356879516125\\4594603035792930665100 6711\end{array}}{6}$$

◦ Or perhaps not – if the Bernoulli number program had been meant for serious use!

Technicalities: deriving the Bernoulli calculation

"... our object is not simplicity or facility of computation, but the illustration of the powers of the engine ..."

The odd (B_{2n-1}) Bernoulli numbers are characterised by:

$$0 = -\tfrac{1}{2} \cdot \tfrac{2n-1}{2n+1} \; + B_1 \left(\tfrac{2n}{2}\right) \; + B_3 \left(\tfrac{2n(2n-1)(2n-2)}{2 \cdot 3 \cdot 4}\right) \; + B_5 \left(\tfrac{2n(2n-1) \cdots (2n-4)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}\right) + \cdots + B_{2n-1} \qquad (8)$$

rewritten as

$$0 = A_0 + A_1 B_1 + A_3 B_3 + A_5 B_5 + \cdots + B_{2n-1}$$

"A_1, A_3, $etc.$, *being those functions of* n *which respectively belong to* B_1, B_3, $etc.$"

In modern notation:

$$0 = A_0(n) + A_1(n)B_1 + A_3(n)B_3 + A_5(n)B_5 + \cdots + B_{2n-1} \qquad (8)$$

and the note proposes:

... we may derive from it the numerical value of every Number of Bernoulli in succession, from the very beginning, ad infinitum, by the following series of computations:

1st Series. Let $n = 1$, and calculate (8) for this value of n. The result is B_1.

2nd Series. Let $n = 2$. Calculate (8) for this value of n, substituting the value of B_1 just obtained. The result is B_3.

3rd Series. Let $n = 3$. Calculate (8) for this value of n, substituting the values of B_1, B_3 before obtained. The result is B_5.

And so on, to any extent.

$$B_{2n-1} = -\left(\quad A_0(n) + A_1(n)B_1 + A_3(n)B_3 + A_5(n)B_5 + \cdots + A_{2n-3}(n)B_{2n-3} \quad \right)$$

$$\begin{aligned}
A_0(n) &= -\tfrac{1}{2} \cdot \tfrac{2n-1}{2n+1} \\
A_1(n) &= \tfrac{2n}{2} \\
A_3(n) &= A_1(n) \cdot \tfrac{2n-1}{3} \cdot \tfrac{2n-2}{4} \\
A_5(n) &= A_3(n) \cdot \tfrac{2n-3}{5} \cdot \tfrac{2n-4}{6} \\
A_7(n) &= A_5(n) \cdot \tfrac{2n-5}{7} \cdot \tfrac{2n-6}{8} \\
&\vdots
\end{aligned}$$

$$\underbrace{V_1 \equiv 1,\, V_2 \equiv 2,\, V_3 = n(\text{initially } 1)}_{\text{Data Columns}} \qquad \underbrace{V_{21} = B_1,\, V_{22} = B_3, \ldots V_{20+n} = B_{2n-1}}_{\text{Result Columns}}$$

$$B_1 = -(A_0(1)) \qquad A_0(n) = -\tfrac{1}{2} \cdot \tfrac{2n-1}{2n+1}$$

1 :	$V_2 \times V_3$	$\rightarrow$	V_4, V_5, V_6	$= 2n$
2 :	$V_4 - V_1$	$\rightarrow$	V_4	
3 :	$V_5 - V_1$	$\rightarrow$	V_5	
4 :	$V_4 \div V_5$	$\rightarrow$	V_{11}	$= \frac{2n-1}{2n+1}$
5 :	$V_{11} \div V_2$	$\rightarrow$	V_{11}	$= \frac{1}{2} \cdot \frac{2n-1}{2n+1}$
6 :	$V_{13} - V_{11}$	$\rightarrow$	V_{13}	$= A_0(n)$
7 :	$V_3 - V_1$	$\rightarrow$	V_{10}	$= n - 1$
☞ 24 :	$V_{20+n} - V_{13}$	$\rightarrow$	V_{20+n}	store B_{2n-1}
25 :	$V_1 + V_3$	$\rightarrow$	V_3	increment n

Mistakes corrected on lines 4 and 24.

Number of Operation	Nature of Operation	Variables acted upon	Variables receiving results	Indication of change in the value on any Variable	Statement of Results	Data $^1V_1$	Data $^1V_2$	Data $^1V_3$	Working Variables $^0V_4$	$^0V_5$	$^0V_6$	$^0V_7$	$^0V_8$	$^0V_9$	$^0V_{10}$	$^0V_{11}$	$^0V_{12}$	$^0V_{13}\cdots$	Result Variables $^1V_{21}$	$^1V_{22}$	$^1V_{23}$	$^0V_{24}\cdots$
						○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
						0 0 1	0 0 2	0 0 4	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0	B in a dec. fract.	B in a dec. fract.	B in a dec. fract.	0 0 0
						[1]	[2]	[n]	[ ]	[ ]	[ ]	[ ]	[ ]	[ ]	[ ]	[ ]	[ ]	[ ]	[B_1]	[B_3]	[B_5]	[B_7]
1	×	$^1V_2 \times {}^1V_3$	$^1V_4, {}^1V_5, {}^1V_6$	$\{^1V_2 = {}^1V_2;\ {}^1V_3 = {}^1V_3\}$	$= 2n$	....	2	n	$2n$	$2n$	$2n$											
2	−	$^1V_4 - {}^1V_1$	$^2V_4$	$\{^1V_4 = {}^2V_4;\ {}^1V_1 = {}^1V_1\}$	$= 2n - 1$	1	....	....	$2n-1$													
3	+	$^1V_5 + {}^1V_1$	$^2V_5$	$\{^1V_5 = {}^2V_5;\ {}^1V_1 = {}^1V_1\}$	$= 2n + 1$	1	....	....	....	$2n+1$												
4	÷	$^2V_4 \div {}^2V_5$	$^1V_{11}$	$\{^2V_5 = {}^0V_5;\ {}^2V_4 = {}^0V_4\}$	$= \frac{2n-1}{2n+1}$	....	....	....	0	0	....	....	....	....	....	$\frac{2n-1}{2n+1}$						
5	÷	$^1V_{11} \div {}^1V_2$	$^2V_{11}$	$\{^1V_{11} = {}^2V_{11};\ {}^1V_2 = {}^1V_2\}$	$= \frac{1}{2} \cdot \frac{2n-1}{2n+1}$	....	2	....	....	....	....	....	....	....	....	$\frac{1}{2} \cdot \frac{2n-1}{2n+1}$						
6	−	$^0V_{13} - {}^2V_{11}$	$^1V_{13}$	$\{^2V_{11} = {}^0V_{11};\ {}^0V_{13} = {}^1V_{13}\}$	$= -\frac{1}{2} \cdot \frac{2n-1}{2n+1} = A_0$	....	....	....	....	....	....	....	....	....	....	0	....	$= -\frac{1}{2} \cdot \frac{2n-1}{2n+1} = A_0$				
7	−	$^1V_3 - {}^1V_1$	$^1V_{10}$	$\{^1V_3 = {}^1V_3;\ {}^1V_1 = {}^1V_1\}$	$= n - 1 (= 3)$	1	....	n	....	....	....	....	....	....	$n-1$							
8	+	$^1V_2 + {}^0V_7$	$^1V_7$	$\{^1V_2 = {}^1V_2;\ {}^0V_7 = {}^1V_7\}$	$= 2 + 0 = 2$	....	2	....	....	....	....	2										
9	÷	$^1V_6 \div {}^1V_7$	$^3V_{11}$	$\{^1V_6 = {}^1V_6;\ {}^0V_{11} = {}^3V_{11}\}$	$= \frac{2n}{2} = A_1$	....	....	....	....	....	$2n$	2	....	....	....	$\frac{2n}{2} = A_1$						
10	×	$^1V_{21} \times {}^3V_{11}$	$^1V_{12}$	$\{^1V_{21} = {}^1V_{21};\ {}^3V_{11} = {}^3V_{11}\}$	$= B_1 \cdot \frac{2n}{2} = B_1 A_1$	....	....	....	....	....	....	....	....	....	....	$\frac{2n}{2} = A_1$	$B_1 \cdot \frac{2n}{2} = B_1 A_1$	....	B_1			
11	+	$^1V_{12} + {}^1V_{13}$	$^2V_{13}$	$\{^1V_{12} = {}^0V_{12};\ {}^1V_{13} = {}^2V_{13}\}$	$= -\frac{1}{2} \cdot \frac{2n-1}{2n+1} + B_1 \cdot \frac{2n}{2}$	....	....	....	....	....	....	....	....	....	....	....	0	$\left\{-\frac{1}{2} \cdot \frac{2n-1}{2n+1} + B_1 \cdot \frac{2n}{2}\right\}$				
12	−	$^1V_{10} - {}^1V_1$	$^2V_{10}$	$\{^1V_{10} = {}^2V_{10};\ {}^1V_1 = {}^1V_1\}$	$= n - 2 (= 2)$	1	....	....	....	....	....	....	....	....	$n-2$							
13	−	$^1V_6 - {}^1V_1$	$^2V_6$	$\{^1V_6 = {}^2V_6;\ {}^1V_1 = {}^1V_1\}$	$= 2n - 1$	1	....	....	....	....	$2n-1$											
14	+	$^1V_1 + {}^1V_7$	$^2V_7$	$\{^1V_1 = {}^1V_1;\ {}^1V_7 = {}^2V_7\}$	$= 2 + 1 = 3$	1	....	....	....	....	....	3										
15	÷	$^2V_6 \div {}^2V_7$	$^1V_8$	$\{^2V_6 = {}^2V_6;\ {}^2V_7 = {}^2V_7\}$	$= \frac{2n-1}{3}$	....	....	....	....	....	$2n-1$	3	$\frac{2n-1}{3}$									
16	×	$^1V_8 \times {}^3V_{11}$	$^4V_{11}$	$\{^1V_8 = {}^0V_8;\ {}^3V_{11} = {}^4V_{11}\}$	$= \frac{2n}{2} \cdot \frac{2n-1}{3}$	....	....	....	....	....	....	....	0	....	....	$\frac{2n}{2} \cdot \frac{2n-1}{3}$						
17	−	$^2V_6 - {}^1V_1$	$^3V_6$	$\{^2V_6 = {}^3V_6;\ {}^1V_1 = {}^1V_1\}$	$= 2n - 2$	1	....	....	....	....	$2n-2$											
18	+	$^1V_1 + {}^2V_7$	$^3V_7$	$\{^2V_7 = {}^3V_7;\ {}^1V_1 = {}^1V_1\}$	$= 3 + 1 = 4$	1	....	....	....	....	....	4										
19	÷	$^3V_6 \div {}^3V_7$	$^1V_9$	$\{^3V_6 = {}^3V_6;\ {}^3V_7 = {}^3V_7\}$	$= \frac{2n-2}{4}$	....	....	....	....	....	$2n-2$	4	....	$\frac{2n-2}{4}$	....							
20	×	$^1V_9 \times {}^4V_{11}$	$^5V_{11}$	$\{^1V_9 = {}^0V_9;\ {}^4V_{11} = {}^5V_{11}\}$	$= \frac{2n}{2} \cdot \frac{2n-1}{3} \cdot \frac{2n-2}{4} = A_3$	....	....	....	....	....	....	....	....	0		$\left\{\frac{2n}{2} \cdot \frac{2n-1}{3} \cdot \frac{2n-2}{4}\right\} = A_3$						
21	×	$^1V_{22} \times {}^5V_{11}$	$^2V_{12}$	$\{^1V_{22} = {}^1V_{22};\ {}^0V_{12} = {}^2V_{12}\}$	$= B_3 \cdot \frac{2n}{2} \cdot \frac{2n-1}{3} \cdot \frac{2n-2}{4} = B_3 A_3$	....	....	....	....	....	....	....	....	....	....	[0]	$B_3 A_3$	....	....	B_3		
22	+	$^2V_{12} + {}^2V_{13}$	$^3V_{13}$	$\{^2V_{12} = {}^0V_{12};\ {}^2V_{13} = {}^3V_{13}\}$	$= A_0 + B_1 A_1 + B_3 A_3$	....	....	....	....	....	....	....	....	....	....	....	0	$\{A_0 + B_1 A_1 + B_3 A_3\}$				
23	−	$^2V_{10} - {}^1V_1$	$^3V_{10}$	$\{^2V_{10} = {}^3V_{10};\ {}^1V_1 = {}^1V_1\}$	$= n - 3 (= 1)$	1	....	....	....	....	....	....	....	....	$n-3$							
					Here follows a repetition of Operations thirteen to twenty-three																	
24	+	$^0V_{24} - {}^4V_{13}$	$^1V_{24}$	$\{^4V_{13} = {}^0V_{13};\ {}^0V_{24} = {}^1V_{24}\}$	$= -B_7$	....	....	....	....	....	....	....	....	....	....	....	....	0	....	....	....	B_7
25	+	$^1V_1 + {}^1V_3$	$^1V_3$	$\{^1V_1 = {}^1V_1;\ {}^1V_3 = {}^1V_3\}$; $^5V_6 = {}^0V_6$; $^5V_7 = {}^0V_7$	$= n + 1 = 4 + 1 = 5$; by a Variable-card. by a Variable-card.	1	....	$n+1$	....	....	0	0										

Some evidence for our conjecture: Lovelace explains Operation 7

Operation 7 will be unintelligible, unless it be remembered that if we were calculating for n = 1 instead of n = 4, Operation 6 would have completed the computation of B_1 *tself; in which case the engine, instead of continuing its processes, would have to put* B_1 *on* V_{21}*; and then either to stop altogether, or to begin operations 1,...7 all over again for value of n(= 2), in order to enter on the computation of* B_3*; (having, however, taken care, previous to this recommencement, to make the number on* V_3 *equal to two, by the addition of unity to the former* n *(= 1) on that column).*

Now Operation 7 must either bring out a result equal to zero (if $n = 1$*) ; or a result greater than zero, as in the present case; and the engine follows the one or the other of the two courses just explained, contingently on the one or the other result of Operation 7.*

In order fully to perceive the necessity of this experimental operation, it is important to keep in mind what was pointed out, that we are not treating a perfectly isolated and independent computation, but one out of a series of antecedent and prospective computations.

Starting configuration: $n = 2,\ V_{20+1} = B_1$

$$B_3 = -\left(A_0(2) + A_1(2)B_1\right) \qquad A_1(n) = \tfrac{2n}{2}$$

Calculemus!

$1-7:$				$V_{13} = A_0(n),\ V_6 = 2n$	In general
$8:$	$V_2 + V_7$	$\rightarrow$	V_7	$= 2$	
$9:$	$V_6 \div V_7$	$\rightarrow$	V_{11}	$= A_1(n)$	
$10:$	$V_{20+(n-1)} \times V_{11}$	$\rightarrow$	V_{12}	$= A_1(n)B_1$	
$11:$	$V_{12} + V_{13}$	$\rightarrow$	V_{13}	$= A_0(n) + A_1(n)B_1$	
$12:$	$V_{10} - V_1$	$\rightarrow$	V_{10}	$= n - 1$	
☞ $24:$	$V_{20+n} - V_{13}$	$\rightarrow$	V_{20+n}		store B_{2n-1}
$25:$	$V_1 + V_3$	$\rightarrow$	V_3		increment n

third (and subsequent) series of computations

13	$-$	${}^{1}V_{6} - {}^{1}V_{1}$	${}^{2}V_{6}$	${}^{1}V_{6} = {}^{2}V_{6}$; ${}^{1}V_{1} = {}^{1}V_{1}$
14	$+$	${}^{1}V_{1} + {}^{1}V_{7}$	${}^{2}V_{7}$	${}^{1}V_{1} = {}^{1}V_{1}$; ${}^{1}V_{7} = {}^{2}V_{7}$
15	$\div$	${}^{2}V_{6} \div {}^{2}V_{7}$	${}^{1}V_{8}$	${}^{2}V_{6} = {}^{2}V_{6}$; ${}^{2}V_{7} = {}^{2}V_{7}$
16	$\times$	${}^{1}V_{8} \times {}^{3}V_{11}$	${}^{4}V_{11}$	${}^{1}V_{8} = {}^{0}V_{8}$; ${}^{3}V_{11} = {}^{4}V_{11}$
17	$-$	${}^{2}V_{6} - {}^{1}V_{1}$	${}^{3}V_{6}$	${}^{2}V_{6} = {}^{3}V_{6}$; ${}^{1}V_{1} = {}^{1}V_{1}$
18	$+$	${}^{1}V_{1} + {}^{2}V_{7}$	${}^{3}V_{7}$	${}^{2}V_{7} = {}^{3}V_{7}$; ${}^{1}V_{1} = {}^{1}V_{1}$
19	$\div$	${}^{3}V_{6} \div {}^{3}V_{7}$	${}^{1}V_{9}$	${}^{3}V_{6} = {}^{3}V_{6}$; ${}^{3}V_{7} = {}^{3}V_{7}$
20	$\times$	${}^{1}V_{9} \times {}^{4}V_{11}$	${}^{5}V_{11}$	${}^{1}V_{9} = {}^{0}V_{9}$; ${}^{4}V_{11} = {}^{5}V_{11}$
21	$\times$	${}^{1}V_{22} \times {}^{5}V_{11}$	${}^{0}V_{12}$	${}^{1}V_{22} = {}^{1}V_{22}$; ${}^{0}V_{12} = {}^{2}V_{12}$
22	$+$	${}^{2}V_{12} + {}^{2}V_{13}$	${}^{3}V_{13}$	${}^{2}V_{12} = {}^{0}V_{12}$; ${}^{2}V_{13} = {}^{3}V_{13}$
23	$-$	${}^{2}V_{10} - {}^{1}V_{1}$	${}^{3}V_{10}$	${}^{2}V_{10} = {}^{3}V_{10}$; ${}^{1}V_{1} = {}^{1}V_{1}$

24	$+$	${}^{4}V_{13} + {}^{0}V_{24}$	${}^{1}V_{24}$	${}^{4}V_{13} = {}^{0}V_{13}$; ${}^{0}V_{24} = {}^{1}V_{24}$
25	$+$	${}^{1}V_{1} + {}^{1}V_{3}$	${}^{1}V_{3}$	${}^{1}V_{1} = {}^{1}V_{1}$; ${}^{1}V_{3} = {}^{1}V_{3}$; ${}^{5}V_{6} = {}^{0}V_{6}$; ${}^{5}V_{7} = {}^{0}V_{7}$

$$B_{2n-1} = -(A_0(n) + A_1(n)B_1 + \cdots + A_{2n-3}(n)B_{2n-3})$$

Hereafter (*i.e.* for n=3, 4, ...) the table can compute any subsequent B_{2n-1}.

Lovelace establishes that her table after line 12 and before line 23 is an *unrolling* ($n-2$ times) of the **repeat** in the following:

$n2$	$:= 2n$	Line 1
a_0	$:= -\frac{1}{2}(\frac{n2-1}{n2+1})$	
a	$:= \frac{n2}{2}$	$a \equiv V_{11}$
sum	$:= a_0 + aB_1$	$sum \equiv V_{13}, B_1 \equiv V_{21}$
$j, k, count$	$:= 1, 2, n-2$	
repeat		Line 13
a	$:= a \times \frac{n2-1}{k+1} \times \frac{n2-2}{k+2}$	Bug corrected
sum	$:= sum + a \times V_{(21+j)}$	
$n2$	$:= n2 - 2$	
$j, k, count$	$:= j+1, k+2, count-1$	Line 23
until $count = 0$		
$V_{(21+j)}$	$:= -sum$	Line 25

It is interesting to observe, that so complicated a case as this calculation of the Bernoullian Numbers, nevertheless, presents a remarkable simplicity in one respect; viz. that during the processes for the computation of millions of these Numbers, no other arbitrary modification would be requisite in the arrangements, excepting the above simple and uniform provision for causing one of the data periodically to receive the finite increment unity.

Lovelace

Conclusion

- The algorithm Lovelace was striving to describe is correct in all but two (small) mistakes.
- Had the Engine existed Lovelace would have noticed the mistakes.

> *It is desirable to guard against the possibility of exaggerated ideas that might arise as to the powers of the Analytical Engine.*
>
> *In considering any new subject, there is frequently a tendency, first, to overrate what we find to be already interesting or remarkable; and, secondly, by a sort of natural reaction, to undervalue the true state of the case, when we discover that our notions have surpassed those that were really tenable.*
>
> Lovelace, [3] Note G

Notes

Note 1: No conditionals? 5 ☞

This remark is from the Wilkes-Bromley correspondence [4]. If Bromley means that Babbage had had no ideas about conditionals and iteration then it contradicts my understanding of the Lovelace-Menabrea paper, in which (a) Menabrea offers a vague description of the function of a counting mechanism to support iteration, and in which (b) Lovelace discusses repeated, even nested, cycles of operation cards cards. If it means that Babbage was unable to provide an implementation of these ideas then it may well be accurate.

Note 2: Wilkes's pessimism 5 ☞

Wilkes was pessimistic about the Engine during this correspondence – as pessimistic as he had been when writing his memoir marking Babbage's bicentennial[5].

From: Maurice Wilkes To: Allan Bromley Date: July 20th 2000

```
...

Are you now suggesting that Babbage failed to
see the significance of what he had achieved?  To
us, he appears to have designed a general-purpose
computer.  To him, the Analytical Engine was a
device that would do a small number of tasks, ...
a multi-purpose machine, not a general-purpose
machine.

I sometimes think that Babbage's scheme of sepa-
rate cards for operations and [variables] would
not really have worked and that in some way Bab-
bage realized this.  Perhaps he tried ...  and
could not get off the ground.

This all suggests that a critical examination
of programming with separate operation cards and
variable cards would be in order.  It might lead
to a quasi-mathematical ‘‘proof’’ of the impossi-
bility of setting up a computing service on that
basis.
```

I don't think Wilkes's quasi-mathematical proof will ever appear: for if mechanisms can be invented (in Babbage's spirit) that implement conditionals, iteration, and the use of computed numbers to denote variable columns then (surely) a mechanism can be invented that drives the variable card drive backward and forward "in sync" with the operation stream. One potential obstacle to this would be if there were no way of determining from an operation how many variable cards to expect it to consume. If the mill's calculation is triggered by the arrival of a result variable card, and could be re-triggered by the arrival of another such card, this would provide such an obstacle.

```
...

P.S. One of the things that I am afraid of is
that one day someone will build an analytical
engine, or more likely simulate one, along Bab-
bage's lines, but with a modern instruction set.
This will be widely accepted by the world as a
genuine implementation of Babbage's engine.

Once the above rabbit has been pulled out of a
hat, nothing that people like us say will be lis-
tened to. Is there anything that we can do in
advance to cut the ground from below the con-
juror's feet?
```

This startling postscript made me wonder what Wilkes's motive could possibly have been, unless it was to ensure that only the *letter* of Babbage's work on the Engine could be used.

Note 3: 7 ☞

Indeed you couldn't have listened to it: the machinery was never completed.

Note 4: 8 ☞

Lovelace emphasises the distinction between mechanism and function in a long and striking passage in Note A. It starts by discussing the difference between Lardner's account of the Difference Engine [2] and Menabrea's:

> *The writer of the article we allude to has selected as his prominent matter for exposition, a wholly different view of the subject from that which M. Menabrea has chosen. The former chiefly treats it under its mechanical aspect, entering but slightly into the mathematical principles of which that engine is the representative, but giving, in considerable length, many details of the mechanism and contrivances by means of which it tabulates the various orders of differences.*
>
> *M. Menabrea, on the contrary, exclusively developes the analytical view; taking it for granted that mechanism is able to perform certain processes, but without attempting to explain how; and devoting his whole attention to explanations and illustrations of the manner in which analytical laws can be so arranged and combined as to bring every branch of that vast subject within the grasp of the assumed powers of mechanism.*

Note 5: 8 ☞

A later pioneer of computing, R.A. (Tony) Brooker, used to say that there were that there were two fundamentally different approaches toward computers and programming: he called their adherents *primitives* and *space-cadets*. One reading of this places those preoccupied with mechanism in the camp of the primitives, and those preoccupied with function and meaning in the camp of the space cadets. Brooker himself was a polymath, one of the role models in computing who helped modern programmers understand that in some phases of a development one must act as if one were a primitve, and in others one must act as a space cadet.

Note 6: 9 ☞

The mechanisms referred to here respectively interpret the operation cards and the variable/number cards.

I know of no 20th century computer or programmable calculator that specifies operations using a completely different mechanism to that which it uses for specifying their operands. Even stack-based machines have load-to-stack and store-from-stack instructions one of whose operands is specified in the instruction stream.

Note 7: Separate Operation and Variable streams 10 ☞

- Motivations: speed-up of the engine, economy in preparing op'n cards
- Effects: Maurice Wilkes and Allan Bromley believed that this was a detail that prevented the implementation of a coherent programing model for iteration. As Wilkes wrote to Bromley [4]

I sometimes think that Babbage's scheme of separate cards for operations and numbers would not really have worked and that in some way Babbage realized this. Perhaps, he tried to write programs using the user interface that I referred to above, and could not even get off the ground. But then, why could he write microprograms? Perhaps because, at the microprogramming level, there are no variable cards or operation cards – in fact, no cards at all.

This all suggests that a critical examination of programming with separate operation cards and variable cards would be in order.

Note 8: 11 ☞

There is a recurring theme in the paper that could well have led to misunderstanding by contemporaries, namely that the *essence* of a specific calculation lies in the operation cards, and that the variable cards are (almost) incidental.

Note 9: 14 ☞

Does this conflate general iteration and Menabrea's multiply-invoked single operations?

$$7(\times) \text{ computes } a \cdot x^7 \rightarrow V_4 \text{ not } a \cdot x^n \rightarrow V_4$$

$$34(\times) \text{ computes } x^{5\times 7} \rightarrow V_4 \text{ not } x^{a\times n} \rightarrow V_4$$

Menabrea's original description (computing b^n in $n-1$ multiplications) of iteration is clearer:

When the number n has been introduced into the machine, a card will order a certain registering-apparatus to mark $(n-1)$, and will at the same time execute the multiplication of b by b. When this is completed, it will be found that the registering-apparatus has effaced a unit, and that it only marks $(n-2)$; while the machine will now again order the number b written on the column V_1 to multiply itself with the product b^2 written on the column V_3, which will give b^3. Another unit is then effaced from the registering-apparatus, and the same processes are continually repeated until it only marks zero. Thus the number b^n will be found inscribed on V_3

His allusion to conditional execution appears soon afterwards

For this purpose, the cards may order $m+q$ and $n+p$ to be transferred into the mill, and there subtracted one from the other; if the remainder is nothing, as would be the case on the present hypothesis [ie. $m+p=n+q$], the mill will order other cards ... to bring to it the coefficients Ab and Ba, that it may add them together and give them in this state as a coefficient for the single term $x^{n+p}=x^{m+q}$

Note 10: 15 ☞

Operation card re-use appears here explicitly – as does a column-reference notation that is not used in its full generality anywhere in the sketch.

Note 11: 18 ☞

- No op-card re-use
- Working-variable columns allocated
- Read-and-clear distinguished from read-and-refresh
- Progress-tracing notation $({}^{i}V_j)$ denotes i^{th} value on j^{th} column
- But details of the variable cards for an operation are now implicit

Note 12: 24 ☞

By the time we reach note G we have seen

- An intelligible notation for planning calculations

  (abstracted away from the details of operation-card-stream compaction)

- A discussion of cycles (loops) in principle (allusions to machinery for "backing" the cards)

But not

- Subscripting: a way of using a calculated value to select a column's value as an operand.

  "*The column numbers are really their names; not their addresses.*" (Wilkes)

Note 13: 30 ☞

I don't quite understand the need for lines 8 and 9 or the use of V_{11} to calculate $A_1(n)$; since the simplification $A_1(n) = \frac{2n}{2} = n$ could have been applied, and n is sitting in V_3.

The transfer of the constant 2 to V_7 by adding V_2 to it in line 8 may have been intended to demonstrate the technique of transferring from one register to another, already zero, register by adding the first to the second. V_7 plays an important role in subsequent series of computations.

Note 14: 32 ☞

The variable j isn't represented as a column in the Engine; it is a conceptual convenience for us when we argue about unrolling the repeat; a so-called *ghost* variable.

The column references $V_{(21+j)}$ are also, perforce, ghostly. But we have experimented successfully with a model of the Engine in which loops and tests are implemented, and in which computed column numbers may be used (indexing). Her table requires almost no attention before Lovelace's unrolled repeat no longer needs unrolling.

References

[1] D. E. Knuth and Luis Trabb Pardo. The Early Development of Programming Languages. *Computer Science Department, Stanford University STAN-CS-76-562*, 1976.

[2] Dr. [Dionysius] Lardner. Babbage's Calculating Engine. From the "Edinburgh Review," July, 1834, No. cxx. In Henry P. Babbage, editor, *Babbage's Calculating Engines*, pages 51–82. Cambridge University Press, 2010. Cambridge Books Online.

[3] L.F. Menabrea. Sketch of the Analytical Engine invented by Charles Babbage with notes upon the memoir by the translator, Ada Augusta, Countess of Lovelace. https://www.fourmilab.ch/babbage/sketch.html.

[4] Allan Bromley and Maurice Wilkes. Events and Sightings: emails between Allan Bromley and Maurice Wilkes. In David Alan Grier, editor, *IEEE Annals of the History of Computing*. 2004.

[5] M.V Wilkes. Babbage as a Computer Pioneer. *Historia Mathematica*, 4(4):415 – 440, 1977.