Title	*010 Transformation of Kets, Continuous and Discrete Transformations and the Rotation Operator*
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Contributor Okay, good morning. So the next item on the synopsis on the website says motion of a particle in a magnetic field. But I think it's better that we postpone that we don't need to handle it now and now I open this new topic, this new chapter which is Chapter 4 in the book the relationship between transformations and observables. We'll come back to the magnetic field later but we have this week and this is – I'd like to make sure we do this thing properly.

Now this topic is off syllabus, right. But it is actually very important it's at the core of quantum mechanics and it's the core of 20th century physics and I think you'll find it illuminating because we now – we – it should explain why the time dependent Schrodinger equation takes the form that it does you should explain why the momentum operator takes the form that it does. Why the canonical commutation relations take the form that they do. So I think it explains many things but for historical reasons it's not actually on the syllabus.

Okay so we have, we know that a – this thing is a function. This thing is a function of x where x is now going to be a position vector this is being the amplitude to find your system, your particle whatever at the location x, right. So it's because it's an amplitude which depends on X it's a complex valued function of X. So we can Taylor series expand as physicists always assume you can Taylor series expand everything. So we Taylor series expand this and we say that if we evaluate this at X minus A then that is going to be essentially – well to keep the notation simple we call this upsi of X, right. So this is going to be upsi of X minus A.D by D X of upsi plus A.D by D X upsi squared over 2 factorial minus blah blah blah. This is for just Taylor series expansion in three variables it's covered in some prelims course, right. And we've – yes.

So this is just the Taylor series expansion and we now make an observation that this can be written with our now, now that we understand how to take a function of an operator and we realise that DBD X is an operator we can write this as E or the exponential of minus A.D by DX operating on upsi. Where we're defining this exponential to mean this thing raised to the noughth power namely one plus this thing raised to the first power plus this thing raised to the second power on two factorial and so on and so on and so forth. That's what we mean by the exponential of this operator okay. But we, we notice that this can also be written as X – this is working upsi of X. This can be written by the rules of operators and the definition of P as the exponential of minus I P upon H Bar A, sorry dot P over H Bar operating on the ket upsi. Let me just remind you what my authority for that is, my authority for that is the observation that or the definition of

P which was that X P upsi was by definition minus I H Bar D by D X of X upsi. And this animal here could be rewritten as X upsi.

Okay so I can just make a change of notation here because of this and where I've replaced P now by the function of P that you see there and then you have a function of this operator. So what have we discovered what we've discovered is that X minus A, this is the bottom line on this little piece of calculation which is only Taylor series expanded into upsi is equal to X on U of A upsi where U of A is the new notation for it simply means the exponential minus I A.P over H Bar. So that's why U of A means. This could also be written as X minus A on upsi is equal to X on upsi primed where upsi primed to ket is by definition U the operator operating on upsi. Right, we just call this thing upsi primed.

So let's think about – so that's just mathematics and it's nothing but Taylor series expanding and a little bit of slight sophistication in taking functions of operators but we've begun to do that we understand that that comes with the territory. What does this physically say? It says that the – so there is if you use this operator on upsi you get a new state upsi primed. What's the point about this new state well if your system is in this state, the new state, then the amplitude to be at X is the same as it was when we were in our old state somewhere behind our current location back at X minus A.

So here's a visualiser, here's meant to be a picture of this if we can get it to come back, yes, so we get it to come back. And if I could find a pointer which I probably can't never mind but so if upsi were that sort of – if the probability density associated with upsi with that spherical blob on the left, the lower left. And A is that vector displacement up there then the amplitude to be at some point – take any point X in the sphere of upsi primed if you move back by A you come to the corresponding point on the spherical density associated with upsi and the amplitude in upsi matches the amplitude in upsi primed that's the sort of visualisation. This statement or what this is telling us, it's telling us that upsi primed, the amplitude of B upsi primed is the same as the amplitude over here at a point back which means upsi primed is the state that our system would be in if we were able to just shove it down the vector A, to translate it by A. Then we would get a new state with these properties. So what have we done? We have discovered what the operator U of A does. U of A shoves the system by displacement A. Now A is just an ordinary boring vector. This is an operator but this is an ordinary boring vector it's a set of three real numbers and we can differentiate we can do D of upsi primed so the place that you – the state that you get is a function of A, right. So we can do D by D upsi primed of A sub I – well A sub J shall we say right to avoid confusion between the index I and the square root of minus 1.

So we can take, we consider the rate at which this thing changes when we change the parameters that appear in here when we differentiate this exponential as everybody knows when you differentiate an exponential you get the exponential back that's just by the magic of that particular power series that defines the exponential. And then we need the differential so that's going to be U, so differentiating U we're going to get back U but we're also going to get the derivative of this with respect to A sub J which is going to be minus I A J, sorry minus PJ over H Bar. And then of course upsi will stick around because upsi is not a function of A.

So if we would now set well so no and now we can just recall that this thing is upsi primed so I now have that D upsi primed by DA sub J is equal to minus I – let's multiple through by I H Bar. And then we have that this is equal to PJ upsi primed – whoops. So this, this now answers a question which I forgot to ask at the beginning of the lecture which is what actually does the operator P do? An operator associated with an observable so with each observables we have associated an operator. We did it originally by saying that Q, the observable associated with Q was by definition QJ QJ QJ. And this operator, we're taking advantage of the fact that omission operation is uniquely characterised by its eigen kets and eigen values. So you specify these, you specify these. If you specify this you specify this. There's a relationship here which we found useful. We've discovered that the expectation value of Q for example is equal to this mathematical animal and other things, and other useful things. We've found the rate of change of expectation values depends on the commutator of Q with the Hamiltonian operator which is the operator associated with the energy etc etc. But we haven't actually addressed or answered the question of what this observable – what these operators that we're introducing actually do to states because an operator turns a state into a new state. So for example – so the operator Q turns upsi if we expand upsi in its eigen states, right, so we – if we write it like this.

So we know we can expand any upsi thus in the eigen states of this operator and then we know how to use this on this so this is equal to the sum QJ QJ up – whoops. So when we use the operator Q on upsi we get this stuff here which is some long gobbly gook. But if we measure Q then upsi goes to QK for sum K. It doesn't go to this long list of stuff it goes to one of these things and the one of these things is chosen at random somehow by nature not discussed by theory, no answer offered by theory merely probability distribution under which we get one of these things is predicted. But we know that the state upsi on making a measurement collapses into one of these states here. So the operator Q is not doing measuring that's the point and we have discovered apperpro of the operator P what is it doing, what P does is give you the rate of change of your state when you shove something along. So PX gives you the rate of change, PX upsi gives you the rate of change PX upsi gives you the rate of change of your state if you shove it down the X axis. So we're learning what the operator PX does and what it does is not measure but displace.

Let's for a piece of practice let's check this out on – let's check this out on this state. Let's for fun apply UA to this state which is the state of definitely being at X and make sure that we can produce X plus A, the state of being at X plus A because if it's true, if you take the state X and you displace it by A you must have this stage, right. Let's make sure that this is the case.

So what we want to do is use UA on X. Now this operator here is a function of the momentum operators, right. It's that exponential A.P. So the natty way to do this is to decompose this into a linear combination of states of well defined P. So we write this as DQP of PXP. So basically I've shoved an identity operator in front of the X. This is a boring complex number what is it it's the complex conjugate of the wave function associated with being – of having well defined momentum. So we know what it is it's E to the minus I P upon H Bar.X over H Bar to the three halves power. We discussed that when we talked about generalisation to three dimensions. That's what this complex number is.

This operator ignores that complex number because it's a linear operator and goes straight to the – to its target which is this then all the P operators in here meet their eigen state P and get transformed simply into their eigen values. So this becomes – when this thing hits this which – the Ps in here are operators but when they meet that because that's it's eigen state they simply become eigen values so we get an E to the minus I A.P over H Bar times the ket, the eigen ket left behind and still we have to do a DQP integration.

Right, so this is no longer an operator because it already worked on that and produced its eigen value. So we can rearrange this, we can put those two exponentials whose arguments are mere complex numbers we can gather them together and this becomes the integral DQP of – over H Bar – sorry that isn't barred that's unbarred, excuse me three house bar – H `[[?? 0:16:53]]` naked constant. E to the minus I P – well let's write it. It doesn't matter what order we write these in you see because this is a number and that's a number. So I'm going to write this as A plus X.P upon H Bar UP. But if I now ask myself what is X in this notation I probably should've written this down originally it was DQP over H three halves power E to the minus I X.P over H Bar P. This, this is just the standard expression which I've essentially used above for decomposing the state of well defined position as the super positions of states of well defined momentum where this is – this thing here is nothing but PX.

So since this is the general formula, this state that we're producing UA on X is given by the same formula but with X replaced by X plus A, right because the only difference between this formula and this formula is here we have an X and there, therefore we have an X. And here we have an X plus A so we should have an X plus A. So this establishes indeed that X plus A is equal to U of A on X. So that's just a particular extra – very vivid example of a basic principle.

So what we want to do now is generalise this to any continuous transformation. We always require proper normalisation. We require upsi upsi is equal to 1. Why? Because this tell us that the total probability to find – to get some measurement, define something is 1 – right that's why we're completely wedded to that normalisation. So we're interested in transformations that preserve this property, this shoving it along transformation was one example in a minute we'll talk about the transformations associated with rotating our system around some axis. But there are many transformations we might make.

So what we require, what we're going to say is that upsi goes to some new fangled state which is some operator U on our old state and the restrict because – in light of this we're going to restrict ourselves to 1 is equal to upsi primed upsi primed. If we take our new states they've got to be properly normalised which means that we're looking at upsi U dagger U upsi – alright? So we require – this is 1 but this is by definition U upsi. So if we take the mod square of this we're looking at that where U is this as yet undetermined operator. And the thing is – so this has to be true for all upsi. For any quantum state this has to be true that this thing is 1 and there's a technical detail about establishing that this is 1. There's a box in chapter 4 of the book doing this which I don't propose to go through it's very straight forward and simple but I don't want to take the time to do it because it's mere mathematics.

From this – from the fact that this has to be 1 for any upsi we can deduce that U dagger U is in fact the identity operator okay from this statement this follows fairly straightforwardly but I'm not actually proving it right now. So operators of this sort as I expect you know from Professor Essler's course are called unitary. So unitary operators are precisely those operators which leave the length of our states unchanged and in the present case for physical reasons the length is 1.

Now let's – so we're dealing with one such earlier on but and let's suppose that U is a function of theta. In that case U is a function of A but let theta just be some parameter where – so theta is a parameter which we can make small. Well shall we say which can go to zero. So the idea is that theta is the amount by which you transformed. There A was the distance which we had displaced so A is analogous to theta. Here we – theta just stands vaguely for the amount by which you can do something and we want to be able to say that we can reduce this amount continuously down to nothing when we're doing absolutely nothing so we're going to have that U of nought is the identity operator because that's the operator that does nothing. So we want to have this parameter. And now we're going to argue that if theta's small we should be able to Taylor expand. I said physicists assume you can Taylor expand everything so we're jolly well going to Taylor expand this so we're going to have that U of theta which is now small is U for theta equals nought which we've said is 1, the identity. And now we're going to write the first order term in a slightly funny way we're going to write minus I theta Tao and then we'll have terms order theta squared.

So this is a Taylor series expansion only the first two terms. The zero term and the first derivative term and all the other terms we've just got wrapped up under order theta squared, not saying what they are. And this is an operator. It has to be an operator because this is an operator – now this of course is an operator, that is a mere number, that is a mere number, a real number. So therefore this has to be doing the operating. But we've just chosen a particular way of writing the first order, the first derivative term in a Taylor series. So this is a Taylor series and it relies only on the idea that theta – that there's a whole family of transformations which could be reduced to the identity transformation as theta goes down to nothing when you don't do anything.

Okay now we want to look at this condition that we want to have a look at the condition that the identity is U dagger U. So let's write U – once U dagger if this is U, U dagger is going to be I dagger which is I and then you'll need the dagger of this which is going to be plus I theta Tao dagger plus order theta squared which we're going to ignore. And that has to be multiplied on I minus I theta Tao plus order theta squared which we're going to ignore.

So when you multiply these two brackets together ginormous job in principle because there're all this infinite number of terms and this and that but we won't need to bother with much algebra. We must get the identity operator and we must get the identity operator completely regardless of what theta is, right. Because this is meant to be this – this is a unitary transformation regardless of theta. So let's work this out – this is equal to the i-, so what do we have, we have the lowest order term is this on this. Then there are first order terms which you get this on this and this on this. So we're going to have plus I theta Tao dagger minus Tao and then we will have terms like this on this which will be order theta squared, this on this will be order theta squared. We'll have this on this which we'll be order theta squared so plus order theta squared. We've accounted for everything through linear order.

So this is supposed to be true for all theta. Doesn't matter what theta we take should be true. If it's going to be true for all theta then we can equate powers of theta on both sides. So the coefficient of theta to the nought, namely the identity, should be the same on both sides. Well it is, that's a relief. The coefficient of theta to the first power should be the same on both sides. On this side of the equation there is, well the coefficient of theta to the first power is nothing so it had better be nothing on this side too. So this implies that theta – that Tao dagger is equal to Tao. That is to say Tao is hermitian. Hermitian operators we suspect are associated with observables. So the argument here is that every such transformation is going to be associated with a hermitian operator and the reason this I was put in here, this was totally gratuitous – sorry right up there. The reason that I was put in there which was a totally gratuitous decoration but it went in because that ensures looking forward it ensures that Tao is a hermitian operator rather than an anti operator which it would've been if the I had not been put in.

So there's a suspicion that this Tao and it will always turn out to be the case, that this Tao will be associated with an observable. This is how observables become associated with operators in both classical mechanics and quantum mechanics or I should've said in quantum mechanics and in classical mechanics. It turns out that it's true in any mechanics.

Right and if we... If we write the equation upsi primed is equal to U of theta times upsi is equal to 1 minus I theta Tao plus dot dot dot upsi and we do D theta primed – sorry D upsi primed by D theta we find that this is equal to minus I Tao upsi plus order delta squared. So if we put theta equal to nought then the delta squared goes away. Sorry the order theta squared and multiply this equation through by I and we get a very important equation which is that I D upsi primed by D theta is equal to Tao. So this observed the operator, the hermitian operator

Tao which we suspect is connected to some observable or will turn out to be connected to some observable in every case is the thing – what does it do? What it does is it measures the rate of change of your states when you change the parameter theta. So this is a generalisation of – where are we? This equation here yes. Alright so this is a concrete example of this.

Now this equation has a tiresome H Bar here why's it got a tiresome H Bar here because in that exponential there's a tiresome H Bar on the bottom, right. So here we had the exponential of minus I A.P over H Bar and if you do the Taylor series expansion of that you get 1 minus I AP over H Bar so that the role of Tao in our conceptual apparatus here is played by P over H Bar there. And it's an unfortunate historical accident that momentum – that this operator which we call the momentum operator has been defined with the wretched H Bar so we have to divide through by H Bar to get rid of what we shouldn't have put in in the first place. So it's one of these many cases in physics where history forces us into a bad notation and even in a degree of intellectual muddle that H Bar would've been better left out but… The reason is that momentum came to Isaac Newton's attention before quantum mechanics or this stuff was thought about and so it came to mean something which is really a derivative thing, which is really something which follows on from momentum's fundamental role which is something which shoves your system, which spatially translates your system. Okay.

And if we want to do – so we've defined Tao, Tao came in here through a formula for U of theta when theta is small. We would like to know how to do U of theta even when theta's large. So for large theta – well what we should say is take a transformation through large theta in N steps. So if we are told to find out what U is for a large value of theta the way to go is to make many transformations one after another through small steps of length theta over N. Then if N is big enough no matter what the value of theta we're given we can write that – what we can do is we can say upsi primed which is U of theta upsi of course is equal to U of theta over N, U of theta over N, U of theta over N. N of these terms all multiplied together operating on upsi. So we make a transformation by an amount theta over N and then another one theta over N – so they're N terms.

And each one of these Us we can use that natty formula up there because for each one of these theta over N is small so this can be written as 1 minus I theta over N Tao plus stuff which we're going to be able to neglect. This is raised to the nth power because N of these terms on upsi and now we take the limit as N goes to infinity to be completely sure that this plus dot dot dot stuff can be neglected, right this plus dot dot dot stuff is order theta over N squared. So to be sure it can be neglected we can go to the limit N to infinity and then we have a theorem of calculus that for what this 1 plus a bit plus something over N raised to the nth power goes to an exponential. So this mathematics now tells us that this is the exponential of minus I theta Tao operating on upsi.

So we introduce Tao as the first order Taylor series term but this apparatus tells us that that's all we need to know in order to find out what U of theta is for any theta which I think's slightly surprising you don't need to know anything in the higher orders. What do we say, we say that Tao is the generator of both the unitary transformations, unitary operator rather and the transformations upsi goes to upsi primed. We say is the generator. By saying it's the generator what this is – sorry this is badly written Tao. But the generator is the operator you stuff in up here in the exponential it's always times minus I for conventional reasons and then a parameter theta that tells you how much you've generated.

So for example P over H Bar, not P sadly but P over H Bar is the generator of translations. That's just jargon. So now let's think about – time to move to a new board – think about rotations. This is where it becomes slightly more interesting because we will discover that in quantum mechanics rotations seem – well they're rather more complicated they seem a bit different from, they are actually significantly different, quite amazingly different from rotations in classical physics. And I think this is not fully understood even now.

Right, so to generate translations we in fact need three operators don't we, we need PX, PY and PZ. Why do we need three operators? Because to define a translation we need to specify a vector because we have to say in what direction we're going to go and how far we're planning to go and those three numbers define a vector or alternatively we can say – well okay you know that. So there we should expect that there are – there's more than one generator of rotations because in order to specify a rotation we have to specify a rotation axis and how far round that axis we're going to go. Right, if you know the axis around which – so here's a solid body I can rotate this in a whole variety of ways to specify one rotation I specify the axis I'm going to rotate around and I specify how far round that axis I'm going to rotate.

So we expect three generators of rotation because we specify a rotation with three numbers. Now there are many ways just as there are many sets of three numbers I can use to specify a translation because I can orient my X, Y and Z axis in any which way I like. There are many ways in which I can specify three numbers that define a rotation and those of you who've done S7 the classical mechanics option will've heard of oiler angles of which there are three, theta fi and upsi. But the handiest way to specify three rotations is actually through a vector we're going to use alpha so that's alpha X, alpha Y, whoops comma alpha Z. So – and alpha hat the unit vector so this now doesn't mean an operator it means a unit vector – whoops unit vector parallel to alpha is axis of rotation. And mod alpha – ouch – the modulus of the vector alpha is the angle through which we plan to rotate by, okay. So these three numbers are a handy, convenient system to specifying which rotation you wish to refer to.

And I now say that there must be – so the rotations form a continuous set of transformations of my system because I can rotate my system by a little bit or a lot and when I – so there must be a state of the system which differs from my previous state only in being rotated. And this state must be reachable by some unitary operator U of alpha. And this apparatus here tells me that U of alpha can be written as an exponential of minus I alpha dot J where this is playing the role of Tao. This is the operator, it's a set of three operators as promised because this means alpha X JX plus alpha Y JY plus alpha Z JZ. So it's a set of three operators JX, JY, JZ they must exist because – and it's going to be hermitian. It's going to be hermitian because this operator's going to be unitary and we've shown the connection between hermitian operators and unitary operators.

So I think this – I hope that that much is absolutely self evident or will be when you think about this through again. There must be this operator, it's going to be hermitian so it's going to be a candidate for an observable and the question arises what observable is it going to be the operator of. The operator associated with translations which had to exist it was a logical necessity that it existed and we have shown that the operator is actually the momentum operator divided by H Bar. So I hope it won't come now as a great surprise that this operator is going to be the angular momentum operator. We're not proving this, I'm saying it will turn out to be angular – oh sorry I'm doing this terrible – angular.

So the angular momentum operators are the generators of rotations in the same way that the momentum operators are the generators of translations but we will – we will have to build confidence that that's the case as we go along. I'm saying that this will turn out to be the case I hope it will be clear. At the moment I just hope that that's a plausible conjecture that it is the angular momentum – that we're talking here about the angular momentum operators. And of course the reason there are three of them is that angular momentum itself is a vector so you can have a angular momentum around the x axis and angular momentum around the Y axis and angular momentum about the Z axis and those – that's because you have those three numbers, you have three operators.

And we're going to have the analogue of – well this formula here is going to be that D by D alpha, the modulus of this angle D by D alpha of upsi primed I times this is going to be the unit vector alpha dot J on upsi. You might want to just check the algebra on this. If you do the derivatives so why is this a function of alpha? Only because this is U, which depends on alpha on upsi which does not. So we're talking about the derivative of this exponential with respect to the modulus of the vector but this exponential could clearly be written as mod alpha times the unit vector alpha do the derivative and you'll find this important relationship here.

So what does the operator alph-, this is just a single operator if you take the unit vector alpha and you dot it into the three operators J you get a linear combination of these three operators which is an operator and what does this thing do for you? It measures the rate of change of your state when you rotate it around that axis that's specified by this, that's what it's physically doing for you. And this is an important relation we'll come back to.

Okay, we're not going to quite finish this but let's get going. So we've talked about translations and rotations and they have this in common that you – they have a free parameter how much you do of them which you can turn right down to zero when you do nothing and they just become – the operators just become the identity. But we have to use in physics important – we have important use to make of transformations which are discreet. You cannot turn them down to nothing you either do it or you or you don't do it. And the classic example is the parity or reflection operator. So if I have an ordinary vector then P turns the position vector X into minus X. There is a transformation you can make where you start with a thing and you choose a point which you call the origin and you move every part of your thing through that origin into another thing. So if the origin is here and my lower hand is here if I move every part of my lower hand through the origin by a certain amount to this equal distance opposite from the origin it should come my upper hand, my right hand, right. Left hands and right hands are related in this way through reflection through the origin. If you put the origin symmetrically between the two. So this is a transformation you can make - this is a mental transformation it's not a real physical transformation but it's a mental transformation. You could ask yourself suppose I had a system which was obtained from my real system right here in the lab by this operation would – I mean a question you can ask is would the dynamics of this system. So as my real system round here is moving around, you know, imagine this thing is a solar system on my hand wiggling would the system that you get above by mirroring each of these points through the origin would that behave like a real thing in the universe. And in classical physics that's the case – this what you see down here reflected through the origin produces a wiggle up here which could happen all on its own. There would be a dynamical system that would produce that wiggling.

One of the amazing – one of the great discoveries of the 20th Century was that that's actually not true in all physics we can direct – I mean when weak interactions are involved things happen – if you make a model by taking your real system and playing this silly game with it you get a thing up here which you can distinguish from a real system because the real system up here couldn't behave in that way. So there is this operation of taking your system and mirroring it through the origin and there's a classical operator P which does this it just changes the sign of all your – of all vectors, right. Of all components of a vector it turns X into minus X. It puts a point to the opposite point across there.

So what's the quantum mechanical analogue of this. Well it's this animal so I'm going to have a long tail on P to imply a classical operator which simply changes the sign, right. This is classical and it's just a change of sign of all vectors. Okay. We want a quantum analogue and the quantum analogue is the thing this I'm going to define it thus. What's this? This is the amplitude to B at X if you're in the state upsi primed – I should've written this separately. This is the amplitude. So P upsi is going to make a new state what state is it going to make this state. What's the point about this state – the point about this state is if you're in this state the amplitude to B at X is minus – sorry is the amplitude to be at minus X if you're in the original state, right.

So this is our original state and the amplitude to be at minus X in this state is equal to the amplitude to be plus X in this state which we've gotten by using P on upsi. So P takes my – the state of my hand here and makes this state, that's what it does. And what can we say about – what interesting statements can we make about this well one very obvious statement that we can make is that if we look at X E squared upsi then that's X P P upsi by definition obviously. We use this rule, we use this rule here on this state so that's equal to minus X on P upsi right because this P and that X using that rule on this state gives me this. I'm replacing upsi in this formula by P upsi and now I can play the game all over again so by using the same rule I find that this equal to minus minus X on upsi which of course is equal to X on upsi.

So if you use – what does this tell me. It tells me the amplitude to B at X when I'm in the state P squared upsi is the same as the amplitude to B at X if I was at upsi. In other words these two states are the same so that implies that P squared is the identity operator which implies that P inverse is equal to P. P is its own inverse and what I should do next is show that P is hermitian but we'll have to do that tomorrow. And therefore P is going to have the e properties of an observable.