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Contributor Okay so let's go. So we were – we just began work on the parity –introduced the parity operator P yesterday. So what does it do? It makes out of your left hand your right hand if you orient it correctly by reflecting, by producing a state which is the same as the state you first had but with everything reflected through the origins. So what you used to find the amplitude to be at minus X is now the amplitude to be at X.

We showed not surprisingly that the square of P is the identity operator because if you reflect something twice you have it back where it was which formally implies that P is P minus 1. And the next item on the agenda is to check that P is a hermitian operator and therefore an observable. And the proof of that is that we take two states Pi and up si and we evaluate this complex conjugate. Let's make sure that is what I – exactly what I plan to do, yes.

So, sorry, yes. So we need to do – what we want to do because we know what P does with an X here on the left of P. We slide an identity operator in between the Fi and the P. So we write this D cubed of X Fi X X P up si. Then we can – oh and we need to star the whole thing because I decided to star the thing on the left, right. That star is that star. X of course is real, integral sign is real.

Then we can use that to replace this so that becomes D cubed X Fi X minus X – oops minus X, minus X up si. And then we need to do the starring operation so that's the integral D cubed X take the complex conjugate of that and it becomes up si – excuse me minus X. And then here we're going to have X and I could write just Fi but just for fun I'm going to write P squared Fi because P squared is the identity operator so that's safe enough. Except I regard P squared as P times P. And then I can take – I can say look this P – this outer P can be got rid of by replacing this by a minus X because I'm using X P some new fangled state Fi primed which is P at Fi. So this can be written as the integral D cubed of X up si minus X minus X P, that's this inner P and outer P's been dealt with by changing that sign on Fi.

And then I can change my variable of integration from X to X primed which is minus X and that's going to produce – that is going to be but this take away the identity operator, sorry that's X primed which is minus X. Take away the identity operator and we're looking at – which says that P since Fi and si are arbitrary that tells me comparing with the initial thing that P dagger is equal to P which we already know is equal to P minus 1. So first of all this says that it's hermitian so it's an observable and this P dagger equals minus 1 so it's also at the same time unitary. So it leaves the lengths of all states the same.

So since it's hermitian so – so what are its Eigen values? We have, if P up si is equal to M up si well this is its Eigen value. Well may be we should call it lamder, more traditional. So if P up si is equal to M up si so it's an ei-, sorry an Eigen ket then we can apply P again and get that P squared of up si which is actually equal to up si because P squared is 1 is also equal to lamder P up si which is lamder squared up si. So we have that up si is lamder squared up si and that applies that lamder squared is one which implies that lamder must be plus and minus one. It has two Eigen values plus and minus 1 and we say that if P up si is equal to up si we say that up si is an even parity state correspondingly of course if P up si is minus up si. So if – that's an Eigen ket with the plus 1 Eigen value. This is an Eigen ket with the minus 1 Eigen value, we say it's an odd parity state.

What does that mean? From what we have up there it means that when you – from the top there the question is… So let's have a look at this. Sorry let's look at the – from the wave function point of view right for even parity we can say that X P up si which is equal to minus X up si by the operations of the P thing but since P up si is equal to up si it's also equal to X up si. In other words the wave function is an even function of X and similarly odd parity states that have wave functions which are odd functions of X etc etc. And when we did the harmonic oscillator we found that for example we found that N is even parity for N an even number. And correspondingly it's odd parity for an odd number. Just as a concrete example.

So we very often classify our states – it's very useful to know whether our states are even parity or odd parity. We like to work with ones that have well defined parity that is to say are Eigen functions of this parity operator. By no means all states are Eigen functions of the parity operator however.

Okay now we do something considerably more interesting which is transformations of operators. So we introduce this – we introduce the displacement operator right last – yesterday. So it was called U of A and it was E to the minus I A.P over H Bar where P is the momentum operator. And we understood that what it did was it made out of a state so up si primed being U of A times up si is a new state of the system, the state that it would have it was identical in all respects except it was shoved along by the vector A. If you shove your system on by the vector A the expectation value of the position obviously has to increment by A, right. So if, so we can make the following statement that the expectation value in the state up si primed of X has to equal the expectation value in the state up si plus A because we have displaced our system. This system is the same as this system except its location has been incremented. It's been moved by the vector A and that is logical necessity.

But this we can write in a different way. This we can write using that expression as up si times U dagger X U times up si. Right, that's just a rewrite of that using this operator here. And this I can rewrite in a different way because I can say this vector A which is just an ordinary boring vector I could multiply by – well I could say that this the following, this is X plus A times the identity operator on up si. Right because it's clear that the expectation value of A times the identity operator is the vector A. So these are equivalent expressions.

So I found that the expectation value of this operator is equal to the expectation value of this operator for any up si whatsoever. And it's shown in a box it's – in the book it's a little box which leads to the conclusion the not surprising conclusion that if that's true that this expectation is equal to this expectation. If two operators have the same expectation for every state whatsoever the two operators have to be equal. So this implies with a little bit of footwork that's relegated to a box in the book that U dagger X U is equal to X plus A where this I is perhaps understood. Well let's put it in because I want that to be an operator, right. This is an operator on the right hand side here.

Right now let's now make – we'll make A small, right. If A is small we know we can expand this in terms of its generator. Right so we can write this as – this thing here can be written as the identity operator minus I A.P over H Bar plus order of A squared. So we've done that before and we're going to do it again. So that becomes the identity operator plus I A.P over H Bar plus dot dot dot which we will ignore times X bracket 1 minus I A.P over H Bar dot dot dot. And that is equal to X plus A, the I can be understood.

So we multiply this all up to find what the terms on the order of A proportional to A are. So A is small but still arbitrary. I mean you could still fiddling around with it but it's small. So this is going to give me the conclusion – this is going to give me I X I is going to equal that. Right, we're going to get an X on the left hand side which will cancel with that and in the next order we're going to get A.P which is small because A is small on X times I. And then we'll have an I times X times minus A.P. So what we're going to be left with in the order of A is I over H Bar A.P comma X. The commutator, because we're going to have this times this and also with a plus sign. And we're going to have this, times this with the minus sign. And what's that equal to? That's equal to A because that's the terms on the order of A on the right hand side. The higher order terms must all cancel, that we leave that to the magic of mathematics and not interested in it.

So we have this relationship here and let's look what this looks like in terms of components. If I look at the – so this is a set of three equations one for the X component of this, one for the Y component of this and one for the Z component of this. So what does it look like? It looks like let's multiply by I over H Bar and swap the order here then this is going to be X comma A.P commutator is equal to I H Bar A. Now let's use – write this out in its components. This is as I say a set of three equations so I can say that X J comma the sum over K of A – well of, it's A J. Sorry it's sum of K, AK PK but I can take the AK outside the commutator because it's a mere number, is equal to I H Bar. I have to write now AJ because this is the – this component in respect to here matches this here right. This is a doc product which is the sum PK AK.

And now I can identify – okay so this has to be true for all small AK. So I can write this right hand side as the sum over K if I want to. Sorry, I H Bar the sum over K of delta J K AK – posh way of writing it. And now I can say because AK is arbitrary the co efficient of AK on the right side has to equal the co efficient of AK on the left side so that leads to the conclusion that XJ comma PK is equal to I H Bar delta JK. So we've recovered the canonical commutation relation between X and P as a consequence of P being the operator which generates translations.

So we've come at this in rather a round about way. Just to review how this has happened. I wrote down a rather arbitrary rule. I introduced P by an arbitrary rule. I said that X P up si is minus I H Bar D by D X, X up si. Using Erenthest theorem I tried to persuade you that this wasn't completely crazy but really it wasn't a very satisfactory job to start in that way. Then we showed that because P has this DBDX structure it is the generator of translations. And as a consequence of its being the generator of translations it must have this commutation rule. And what we should've done really is we should've said "Look there must be some operator which generates translations. This operator is going to have this commutation relation and we should've worked our way down to finding out that in the position representation it's represented like thus. And for the angular momentum operators this is the line of argument we're pursuing. We are using – we're introducing them as the generators of rotations and then we're going to find out what they look like in the position rotation and the position representation later on.

So we've come at this in a slightly tortuous way. This is the main job that momentum – the momentum operator has. It's interesting – so it's probably worthwhile just checking that this commutation relation guarantees that rule up there that U dagger XU is equal to X plus A even when A is big, alright. So we've – this stuff has all been for an infinitessible A and it's good to check that the other thing works. That it sorts us even for A big.

So now let's just talk about for any A including a big one, any big displacement. So we're going to be looking at U dagger – sorry XU let me just check my – yes. Which I can write of course as U dagger UX plus U dagger X comma U. So I've just swapped the order of those two and put in the commutator that compensates. This of course is going to be X because U dagger U is 1 and what's this going to be? So it's going to be X plus U dagger of X comma E to the I minus I P, excuse me, A.P over H Bar close brackets.

So this is a classic example. We studied this problem before; we're doing the commutator of X and a function of P. This is the function of P we're doing it. And do you recall that the answer to that problem was that we could write the commutator as the rate of change of this function with respect to T, P sorry times X comma P. So this can be written as X plus U dagger. The rate of change of this with respect to the – the derivative of this with respect to P which is going to be of course is going to be U because the derivative of an exponential is the exponential times the derivative of what's up here with respect to P. So it's going to be U D by, well DBDP of this is minus I. No – sorry let us do this as the derivative with respect to A.P. Right, we'll regard this as a function of A.P. I'm worried about components and the way I can get out of that is considering this to be a function A.P which is just one thing.

So if I take the derivative of this with respect to A.P I get U because I get the exponential back. And then I have times minus I over H Bar, right that's the derivative of the argument of the exponential with respect to A.P. And now I have to write down the commutator of X with respect to A.P.

So this of course produces 1. And what does this produce? This could be – let's write that down that's X plus – whoops minus because of this I over H Bar. This is producing a 1 and now I need the derivative of this which is the sum over K of X comma PK times AK, right. It doesn't matter what order I put down A because it's a mere number. And this it may be that we ought now to introduce an index on X otherwise we're going to get into a confusion. So let's make that I so I'm making – this was a vector X an arbitrary component. Let me call that component I then this becomes I. Then this becomes delta I H Bar over delta IK. The I and the I make a minus 1 which cancels this. The H Bar kills that so this is equal to XI and then this is nothing. This is nothing as K is sound except when K equals I so that becomes an I. So this just becomes an AI and yes it does sort us. That thing is equal to X plus A as advertised at the top there.

So now let's think about rotations. So we have, we introduced these operators. We had JX JY and JZ so that alpha dot J generated rotation through mod alpha about the unit vector in the direction of alpha. Right that's what we established. Well we used that notation we said there had to be such a thing. And what we want to do now is talk about – is apply – is adapt that argument to this case.

So the thing is the expectation value – so, sorry we've… Let's let up si primed be the state that you get when you use U alpha on up si. So this is the state of the system which is identical to this state except it's been turned round through an angle around the axis as advertised up there, right. So we can say something about the expectation value of X of this system must be the same as the expectation value of that system but rotated to. If you've rotated the system you've rotated the expectation value of X. So we can say that up si primed X up si primed, which is – now this thing is a boring vector, right. It's the expectation value of a vector operator so it is a boring vector. It's a set of three numbers. And a set of three numbers we can use a boring rotation matrix on which I'm going to call R alpha. So this is a three by three rotation matrix. An ordinary boring rotation matrix such as I think you must have studied in Professor Essler's course operating on up si X up si.

So this was the old expectation, the expectation value in our unrotated system. If you rotate that expectation value you must get the expectation value in the rotated system. So that's the analogue for the rotational case of that statement U dagger XU is X plus A. Well except I haven't yet written what this is. So I'm going to write this as up si U dagger of alpha X U of alpha up si, right. So I'm replacing this with that.

And now I am saying that for any state up si this expectation value equals this expectation value. Ergo since this is a set of boring numbers it can go inside the expectation and just rotate the operator. So it's just taking a linear of this thing. This is a boring three by three rotation matrix so if I allow it to go inside there it's simply going to be taking a linear combination of the X Y and Z position operators. So I'm going to be able to say that U dagger X U is equal to R alpha of X. That's a matrix.

And now of course I'm going to express this as 1 plus I alpha dot J dot dot dot. So for – this is now, we're now making small alpha so we're rotating it through a small angle. Then this can be written thus. Here is our X, here is our 1 minus I alpha dot J plus dot, dot, dot. And that is equal to the – to this vector it's a vector of operators but still it's a vector rotated by a small angle. Now we know that it is, well at least I hope we do – come on, come on. Oh no it's gone to sleep do I have to draw? Grr, yes, sorry the system seems to have… No, it's gone to sleep.

So this requires a bit of – this is a piece of just standard geometry. What I want to do is write the action of a rotation matrix when it – for a small angle. If I rotate something through a small angle and I hate drawing these diagrams it's going to be someth- - has it come alive. Oh right, yes, okay it just – it went to sleep and needed warming up. So we're looking at this second diagram can you see it because I jolly well can't it's too faint. Anyway so the point is that this is the vector V here is the rotation axis alpha. We're rotating it through a small angle therefore this distance there is small, the displacement that you have up there. This is the rotate vectors on the right, the unrotated vectors on the left. The displacement is this thing here which is the vector – delta alpha – or the vector. So the small rotation vector crossed with the original V. So that we can say that V primed, the rotated vector is equal to the original vector plus this small rotation vector crossed into V.

So the right side so I'm not going to draw this horrible diagram. The right side. So this is going to become X that's the V up there plus alpha cross X. So our alpha is small so we don't need the delta alpha it's just alpha we've made it small to get rid of symbols. So we do the same old stuff we multiply this out on the left to – up towards alpha. We notice that the I, the X and the I produces an X which cancels with the I on the right. And we find that what we're left towards alpha is alpha dot J times X minus from the I the X and the alpha dot J, the thing the other way around. So we find that I alpha – whoops. Alpha dot J comma X is equal to alpha cross X.

Now we need to write this in – we need to introduce indices in order to disentangle what's going on around here. So this is going to be the sum over K I times the sum over K of alpha K which will come out times JK. Alright that's alpha K JK comma XJ. No this is, sorry let's change that. Let's change that to the sum – the thing we sum over to be J it doesn't matter what we call it but let's call it that. We'll call that K, alright? What's that going to be? That is going to be the K component of this vector on the right. Oh sorry we should call that I, right. This is going to be the I component of the vector on the right. Now a cross product can be written as the sum over J and K of epsilon I J K. This is the thing which keeps changing its sign if you swap any two indices it changes its sign and epsilon 1, 2 and 3 is 1 – which I hope you've met in Professor Essler's course. So this is just writing across product intense note – in cartigan tense of notation. Nothing to do with quantum mechanics it's just standard vector algebra. And we have arranged it so that we have the I component of the left side here and the I component on the right side there.

Now we play our trick of saying that look alpha is arbitrary. It needs to be small but otherwise it's arbitrary. You can choose it's direction any which you like and it's magnitude in detail you can choose in any which way you like so we can compare – we can equate the co efficients on the two sides multiply by through, through both sides by I to get rid of this. You're get a minus sign, swap the order of these in order to clean it up and we will have XI comma JJ is equal to I times that's this I brought across the sum over K. The sum over J will go away because we're equating the co efficient of J on the two sides of epsilon IJKXK. This is a terribly important relation. It tells us how J commutes, how the J component of angular momentum commutes with any component of the position operator. But crucially in this argument here we have used nothing about the position operators except that the components form – the three, the X Y and Z operators are the components of a vector.

So all of this argument could be repeated for the three component operators of any other vector, for example for P. So it follows immediately we've only used only property of X used, of the operator X used is that it's a vector. So we've really shown that for any vector this relationship holds. So we've shown that VI JJ equals I epsilon sum over K of epsilon I J K VK for any vector operator. So a vector operator is just a set of three operators if you like whose components are the – whose expectation values will be the components of some classical vector.

Okay so we can apply – well we can immediately apply this as well as to X to VI is PI, the momentum. And we can also apply it to VI is equal to JI the angular momentum. Why is that?

Male 1 Can you put J and V the other way around? `[[?? 0:33:43]]`

Contributor Oh sorry in which one? This one and this one?

Male 1 Or in J before the X `[[?? 0:33:51]]`.

Contributor I lost a sign somewhere. No I think this is right.

Male1 I know but you multiply 'I'.

Contributor Yes I multiply it through by I and I swap the order of these two didn't I?

Male 1 `[[?? 0:34:11]]`.

Contributor No I don't think so. No but this order is the same as this order surely to goodness.

Male 1 Yes but the other way around.

Contributor Okay, let me take advice on that. I'm. . . yes I can't help being sceptical but I suppose I should look here. I suppose I should look. Yes, true. Yes the thing I'm thinking of – okay so may be I have. Dear, I think we – have I drifted a sign somewhere?

Male 1 [[?? 0:35:11]].

Contributor Well, no, this is definitely the [['I' 0:35:21]] component of that cross product. This is definitely so. Do we agree about that? That's – we should just check whether that is – whether this is ordering is as advertised. This is the closest thing.

I can't see it at the moment I think it's probably – it's incredibly hard to do these sign problems on blackboards. Let me leave that and I will confirm tomorrow what the case is I imagine the book is right and I'm wrong but I do not see where I have made the mistake as things stand. Everything looks respectable. No I can't see… I cannot see an errant sign. That's nasty and it does as you say matter, yes. Yes I'll try and sort that and write up before tomorrow's lecture the way it should be.

We need to be persuaded that this. Well the next thing I want to do is be persuaded that this is – so can we apply this to the angular momentum operators. It's going to be very important that we can. And what's the argument? The argument is that alpha dot J has to be a scaler – why? Because the operator U alpha which is E to the minus I alpha dot J this thing is the rotation around a certain vector. What this operator is – this is a physically, you know, meaningful thing and it is defined not by the three numbers that we happen to use to define the direction of the vector but by exactly what that vector is. If you change your coordinates system, you use a new coordinate system you'll be using a new set of three numbers to define this, right. But you must get the same direction in space and that will be the case if these three operators also transform.

So the new operators, the operator associated with J X primed where X primed is your new X axis will be a linear combination of the old operators, the operators associated with the old axis using the proper rule for a rotation. Then this dot product will stay the same and this operator will stay the same as we require. So this operator will be independent of your coordinates system only if these three things transform amongst themselves as for a vector. So J must be a vector and that means we can use it in here. That is to say, we can say that JI comma JJ commutator is equal to I epsilon sum of K, whoops I sum over K epsilon I J K J K. So this is a crucial relationship. Then from that we will find out what the Eigen values can be of these operators JI and JJ and then we'll be able to find what the states of well defined angular momentum are and everything else. This expression is right, right because it's independent of any swap. Can it be that both expressions are right? Well I can't – I mustn't take time to think about it.

Okay let's consider what's a scaler – let's consider a scaler operators. So what is a scaler operator it's an operator which – well a scaler, sorry, a scaler in ordinary physics is a number whose value is unaffected by a rotation of your coordinates, right. Like a dot product it's unaffected by a rotation of the coordinates. So what can we say is that if S is a scaler operator and we – then the expectation value – the expectation value of a scaler operator between rotated states must be the same as the expectation value between the unrotated states. Because this is a boring number and it's evidently by definition a scaler, something unaffected by rotation so the fact that you've rotated your system shouldn't have any effect.

So when we ask ourselves what does that – what implications they'll have it's that U dagger SU is equal to S. We can multiply on the left by U which is the inverse of U dagger because U is a unitary operator. So we have then that SU is equal to US which means that S comma U equals nought, where this of course is U of alpha the rotation operator throughout. So a scaler operator commutes with this rotation operator and it's easy to see by expanding this as 1 plus – so if we write U is the identity minus I alpha dot J etc that immediately goes to the statement that S comma J I equals nought. So scaler operators commute with all the angular momentum operators.

There's a very important and interesting scaler operator and that's J squared which means the sum over K JK JK AKA also known as J.J, right? That's a scaler operator. Every dot product is a scaler operator so we have statements like J squared comma JI is nought. We have statements like X squared comma JI equals nought. We have statements like P squared comma JI equals noughts. These are all important results that we'll use many times. We have statements like X dot P, that's a scaler operator comma JI equals nought. And so on and so forth. So there are many operators we could make out of the operators already on the table which commute with the – all the angular momentum operators.

So, just the summary now of the angular momentum commutation relations. We've got that JI comma JJ is equal to I sum over K epsilon IJK JK. So the individual components – this is a somewhat strange state of affairs. The individual components of angular momentum do not commute with each other so you can't expect to know simultaneously the angular momentum around the X axis and the angular momentum around the Z axis. In individual cases you can but as a general rule you can't expect to know that so there isn't a complete set of states which are simultaneous Eigen states of JX and JZ for example.

But we do have that J squared comma JI equals zero and therefore there is a complete set of mutual Eigen states of the total angular momentum and the angular momentum along any axis. And that's what one always – what we have to work with that we have to consider states which – we have to work with states which are mutual Eigen states of J squared and usually the axis we choose we have to choose one because of this business. And the axis we usually choose is the Z axis.

An important result about parity. Let's go back to the parity operator now. So in the same spirit if I consider – so the expectation value of X – if I reflect my system through the origin right by using the parity operators I make a system which is like my existing system but reflected through the origin. It's obvious that the reflected system is going to have an expectation value of X which is minus the original expectation value, right. Because you've reflected everything and therefore the – if there was an average value of X of – in the original system the reflected system will have minus that value. So this can be written as up si P dagger X P up si, right because up si primed is by definition P up si, P dagger here. But we know that P dagger is P so we have that the expectation value for any state whatsoever of minus X is equal to the expectation value of P X P. So it follows that minus X is equal to P X P multiplied through by P and use the fact that P squared is equal to 1. And we conclude that PX plus XP is nothing very much.

So you can say now that the parity or P anti commutes this condition with a plus sign there, right. With a minus sign it would be a commutator with a plus sign it's anti commutation, anti commutes with X and in fact with any vector. Right, because this argument here really only exploited the fact that we were talking about a vector not necessarily the position vector.

Now why is this stuff important? The practical importance of this is as follows. Suppose we have a state of well defined parity – okay so let P up si equal either plus or minus up si – don't care which but it's going to be… So up si's a state of well defined parity and we've seen that the Eigen states of the harmonic oscillator Hamiltonian are actually states of well defined parity. And now let's consider up si X up si. Well that we've just seen is equal to minus up si P X P up si. This is a pure rewrite of a line higher up there – well except I've taken the dagger off the P but as we know P dagger is P so who cares. So – but we've acquired a minus sign, that's that minus sign.

But P on up si is equal to either plus or minus up si I mean say plus up si and P on this up si is equal to plus up si. So these two Ps can be got rid of if we put in a couple of plus signs or if P up si's minus we get – we have an extra – we take a minus sign out but then we get another minus sign from there. So either way we're taking out two of some sign and therefore this is definitely one. So we have but it's inevitably the case that this expectation value is equal to minus itself. The only number equal to minus itself is zero so that implies that the expectation value of X vanishes for all states, any – all states of well defined parity.

This is a result we use very often and it doesn't just apply to X it applies to any vector operator, right. I could've made X any vector operator and repeated the argument. So when you're in the state of well defined parity the expectation values of all parity operators are nothing. And I think that's – we've one or two minutes in hand but I think that is the moment to stop because the next section is on symmetries and conservation laws.