Title *012 Angular Momentum and Motion in a Magnetic Field*
Description Twelfth lecture of the Quantum Mechanics course given in Michaelmas Term 2009.
Presenter(s) James Binney
Recording http://media.podcasts.ox.ac.uk/physics/quantum_mechanics/audio/quantum-mechanics12-medium-audio.mp3
Keywords physics, quantum mechanics, mathematics, F342, 1
Part of series *Quantum Mechanics*

Contributor So, yesterday we had an awkwardness about this formula here because whatever I derived disagreed in the ordering here from what's in the book and it just should reassure you that what I derived was correct and what's in the book is correct. Because these are important formulae so it's good to have them in your mind.

So, let's just understand how it comes about that both of these are right. So, what's happened between these two is that these things have been swapped in their order but crucially, also these things have been swapped.

So, if I go from here, I pick up a minus sign. If I simply invert the two operators, J and V – V is just an arbitrary vector, okay? It could be X, it could be P, it could be J, it could be whatever, right? So, if I just swap these two operators of course I pick up a minus sign. But then I can pick up a matching minus sign on the right side by swapping these two indices of the epsilon symbol. So that's minus I, sigma K, epsilon J, I, K, V, K.

And now I can say to myself, well – so I now have this formula. From this formula I've arrives at this formula except that what here is called 'I', which is the index on J and the first index on epsilon, is here called little 'j' and what here is called J, being the middle index on epsilon, and the index on the vector operator, is here called 'I'. So, it's a mere relabeling of what appears at the bottom. So these two formulae are both correct.

Okay, so. You just need to pull together - we really have everything now – we just need to pull together the results that we have and just calmly understand the physical significance of them.

So, we've discovered that these operators, like the momentum and the angular momentum, are associated with displacements. They're the generators of displacements, the momentum generates U of A, which shoves your system – it doesn't literally shove your system – it makes a new system that's translated. It's the same as the old one except its location is being incremented by A. The angular momentum operators make a new system, which is the same as your old system except they've been rotated around the origin by some angle.

And we have already seen that when the momentum operator commutes with the Hamiltonian - when one of these observable commutes with the Hamiltonian – we have a conserved quantity. And we have good quantum numbers and things like that.

Okay, so; so let's say, if P commutes with H, what does that mean? That very easily implies that U of A, the translation operator, is going to commute with H because this operator, right, is E to the minus I, A dot P – on H bar, is a function of P and therefore it commutes – if H commutes with P, it commutes with any function of P. So, if we have – sorry – if this equals nought, then this equals nought. Now, what does that tell us?

To see what it tells us we have to think about unitary operator associated with H because H is an observable and it's associated with the unitary operator. E to the minus I; H upon H bar; T. Which we're going to call U of T. What's this operator? Well, we already know that if `[[psi 0:04:29]]` at time T is equal to the sum AN; E to the minus I; EN over H bar; T times EN, don't we? This is our standard expression for solving – our standard means of solving the time-dependent Schrodinger equation is to decompose the given state into a linear super-position of energy. I instates that at any particular time, for example, at T equals zero, and then evolves by multiplying each term in this series by this exponential here.

But we can see that this could also be written as E to the minus IHT upon H bar; times sum AN EN. Right? Because when this linear operator looks at this sum here it passes – it's a linear operator so it can be distributed, passed through these ANs and look at each one of these things, then H looks at it; `[[I can get 0:05:31]]` EN, and it says, "Aha! That's my I can get!" It returns EN times the number EN, and there you go – you have that.

So, this operator, U of T, is a crucial operator. It's the thing that evolves you forward in time. Any state. This is nothing new; this is just a repackaging of old results. So, the unitary operator associated with the observable time moves you forward in time. It carries you from today into tomorrow. Or whatever, right?

So, let's just repeat this; if P comma H equals nought, then that implies – and indeed is implied by – that the unitary operator, U of A commutes with the unitary operator U of T. Right? This is the thing that moves you forward in time. This is the thing that makes you a new system shoved along the bench by A.

So, what does that tell you? That tells you that take the state of your system and evolve it in time and then shove it along the desk. And you will have exactly the same state as if you take your system, shove it along the bench and then evolve it in time. It says that whether you let it evolve here, and then move it to its point where you want to have it, that will give you the same results as if you move it now and let it evolve over there. So, the physical implication of this simple is that physics is the same here as there. Now that's not always the case.

If that were a clock at we let it evolve on the floor, until tomorrow and then read it – oh, and then moved it up here – well, somewhere higher; up here – we wouldn't have the same situation as if we let it evolve until tomorrow up there. Because clocks – the gravitational potential down on the floor is lower that it is up there. So a clock up there will evolve fast than a clock down there.

So, it is by no means obvious it's not necessarily the case that it doesn't matter where you conduct your experiments. That evolving it in one place and then shoving it and moving it somewhere else is going to give you the same results as shoving it somewhere else now, and then allowing it to evolve in another place.

This being the same here and there is statement about the homogeneity - so when this is the case, it's a statement about the homogeneity of space. And physicists are of the view that ultimately physics has to be same here and there, and the reason that the clock evolves on the floor in a different to on the table is because – not because of any in-homogeneity of space – but the fact that there's a dirty great planet here, or 8,000 miles or whatever it is from the centre of the earth and is the relative movement of the earth and clock which has changed the circumstances; not the in-homogeneity of space.

So, we're completely wedded to the concept that fundamentally space is the same everywhere and therefore, fundamentally, this should be the case if your system is isolated. In other words, we say when this principle is not observed – yes, but the reason it's not observed is that your system, Johnny, isn't isolated. In the case of the clock on floor or where it's obvious what the not-isolated-ness is, it's the dirty great planet. But in other circumstances, it might be more subtle but we would – we conjecture that you will be able to find something which is violating the isolation – which is effecting, which is violating its isolation.

Okay. So where are we? This commuting of operators is associated with something being conserved. That something is momentum. It's also associated with a statement about invariance of physics under translations. So, we have a sort of set of ideas like this commuting – well – P with H – well, no. So P comma H is connected to conserved momentum, which is itself connected to uniformity of space. Which is the same thing as symmetry under translation. And this is a set of three sort of separate things which are tightly connected by mathematics and basic principles of physics.

Similarly, if it's the case that say JZ comma H – so this is the generator of rotations around the Z axis – if that's equal to nought – sorry, I need to have a nought equals here – then that is associated with conservation – in classical physics, that's associate with the conservation of angular momentum. Which is why we want to call this the angular momentum operator, which is associated with the isotropy of space. So it's clearly the case that a compass behaves differently if you orient it east, west or north, south, right? Because on the surface of the Earth on account of the Earth's magnetic field, the physics of space is not isotropic from the perspective of a compass needle and it's associate with – and as a consequence of that – its angular momentum operator will not commute with the Hamiltonian of a compass needle. And its angular momentum will not be conserved, that's why it swings to and fro around the North Pole when you let it go; magnetic north.

Oh, yes; and on the momentum thing, just remember what Newton said about – what is moving in a straight line Etc,. he fundamentally said that isolated bodies have conserved momentum and so there already he was in fact connecting the conservation of momentum to the isotropy of space. He had that concept of an isolated body.

So, in general we're always interested in finding these operators – these observables – which commute with the Hamiltonian and in general it's hard to find – so, in general it's hard to find the operators that we don't have a system unfortunately for finding operators that commute with the Hamiltonian. The best system we've in fact got is to look at the uniformity of the physics to say to ourselves, "Can I see any reason why the system should be different, the behaviour of the system should be different if I rotate it or if I translate it or do some other thing to it?" So, observables commuting with H.

But here's an example of when you can spot one. If you have N particles that interact with each other and nothing else – whoops, interact – then you have that the Hamiltonian of this system is the sum of PI squared over 2N I summed over particles I – so this index here remunerates the particles – plus – so that's the kinetic energy of each particle summed over particles makes the Hamiltonian to the whole system – and to that we have to add the potential energy of interaction between the particles, which will be the sum over pairs of particles which we can get by saying that J is less than I – these are the vector positions of the particles – so there's some interaction potential between these particles, they interact in pairs in this picture. So this will be the Hamiltonian if these particles which were interacting in some arbitrary way with each other. Because they interact in pairs.

And what we can say is that H is- it's invariant. This expression is manifestly the same if XI goes to XI plus A. If you simply add a vector A to all the locations of the particles as long as you shove the whole system along via vector A, then the arguments of all of these interactions stay the same and you don't effect the momentum, so H is invariant.

And what does that tell you? That tells you that the generator of this transformation is going to be a conserved quantity. So this transformation – well, it implies conservation of the generator which is going to be the total momentum being the sum of the momentum of the individual particles. Which of course we recognise as the – the total momentum of this system is going to be conserved because action and reaction are equal and opposite, back to old Isaac Newton.

These points you haven't seen probably made in this way before but I would like to make the point that they are actually very fundamental points of physics. They are not peculiar; they are not special to quantum mechanics. In classical physics all of these statements remain true, it's just that when you do elementary mechanics you don't have machinery at hand to see the connection between symmetry in conserved quantities. These are really very basic points which are true in quantum mechanics, but they're also true in classical physics. But we have no the apparatus so we can now see these things rather more clearly than we can in classical physics.

So, we now have time to cycle back to something I skipped which was –which is motion in a magnetic field. This is a particularly important topic because an awful lot of quantum mechanics was developed – historically it was developed by sticking atoms into magnetic fields. It's obviously also an important topic in the sense that we use magnetic fields in an awful lot of devices and people also now stick their crystals in magnetic fields to see what happens. So, it's still an important way of probing systems when you're trying to understand systems which – whose physics is based on quantum mechanics, and it's very important to understand how this happens.

And there's a fundamental difficulty that we have to address up front which is – what we need to know is how to modify the Hamiltonian. Because in quantum mechanics you put the physics into the Hamiltonian. The Hamiltonian tells you what forces are acting, what the system consists of. It encodes what the physical laws are for your system. So if you switch on a magnetic field it must be that it is changing the Hamiltonian somehow, so the question is, is it changing the Hamiltonian?

And if you take the view that H is equal to P squared over two M plus V – because you've got some particle – then you're in trouble because there's no magnetic contribution in the potential energy. Because the Lorentz force never does any work. The Lorentz force, V cross B in perpendicular to V so V dot V cross B identically vanishes and the Lorentz force never does any work, so it can never contribute to the potential energy of your system and therefore you can't look for magnetic contributions in here.

So, it turns out that because magnetism is relativistic correction to electrostatics, right? Fundamentally, that's what it is and I think it's one of the most – I'm always amazed and I don't think I really understand why it is that our electoral devices overwhelmingly use this – you know, you vacuum cleaners, your disc drive, your… - I mean, we make the electricity in fact using a relativistic correction to electrostatics. We do almost nothing with electrostatics. A few scientific instruments use electrostatic drives but it's almost an unused – you know, coulance is almost always unused – except to get the electrons to go down the wires in order to generate this relativistic correction because they're moving at a slightly different speed from the ions in the wires.

Anyway, so. But it is a relativistic correction to electrostatics. And so in order to find out how to change your Hamiltonian, you really need to do relativity; that's the proper place to look and I'm not going to be able to derive this for you. I'm going to be able to tell you what it is which is – what we need to do is replace that P by P minus the charge on the particle. So when we have a magnetic field, we put it in by replacing P in our original Hamiltonian by P minus QA, where this is the charge on your particle. And of course, B is the curl of A. So A is the magnetic vector potential that generates the magnetic field.

So, I'm not going to be able to justify this because to explain why this should be so, we need to do some relativity which is way out of scope for the quantum mechanics. But what I should do with this is use `[[aerothest 0:22:44]]` theorem to convince you this gives you – that this gives us the classical equations of motion in magnetic field. And ultimately, only experiment can tell you if this is right or wrong.

So, let's use aerothest to recover classical physics out of this. So what we have is IH bar, DBDT – whoops. DBDT – of the expectation value of XI. What's that? That's equal to the expectation value of XI comma H. We'll drop this because we're not really interested. For the moment we're not interested in V which would contain, for example, the electrostatic interactions, the interactions with the electric field if there were any. Let's not worry about it. Let's just take it that what we have is a particle moving in magnetic field so I want to take this to be the Hamiltonian to keep life simple.

So, what is this? This is one over two M, expectation value of XI comma P minus QA squared commutator of psi. Now we know how to take now commutators. Why does XI not commute with this? It doesn't commute with this because it contains the momentum operators – well, in particular it contains the `[[i'th 0:24:16]]` momentum operator. And this is going to be one over two M if we're pedantic – you could do it more quickly than this.

There are two of these coupled together. It'll commute with the first one. We should do the commutator with the first one. Now X is going to commute with A. We're going to have that XI comma A equals nought because this vector potential is a function of X, right? The vector potential depends on X; it varies with X, therefore is a function of the operator X. So, X is going to commute with it.

So, the reason X is going to commute with this bracket is because it contains PI. So, we're going to have XI comma P – the vector P – dotted into P minus QA; close brackets – that's one of the two terms and then unfortunately there's another term which will in fact be identical. But just to be pedantic let's just keep it right. This is going to be P minus QA dot XI comma P.

So, what we've done is regarded this as P minus QA times P minus – dotted into P minus QA which is a product; we've used the rule for using the commutator on a product, do the commutator on the first term, we'll leave the second alone – that's that – then leave the first term alone and do the commutator on the second one. And the commutators on these brackets reduce to me the XIP because of that.

This dot product could be written as a sum over K if PK dot PK minus QA K, right? And this commutator is going to be nothing except when K equals I, when this will be an IH bar which will cancel that IH bar and we'll discover that DBDT of XI is equal to - this IH bar is a mere number, it'll commute with this so we don't need to – this term is going to generate the same as that, so this gives me a one over M of P minus PI minus Q AI.

What this is telling us is that the classical velocity, because the rate of change of the expectation value, the position is what we would call the classical velocity – the expectation value if you like – is not equal to the momentum over M; it's equal to the momentum minus QA over M. Or alternatively, it's telling you that PI is equal to M VI – oh, sorry. They're expectations values here. They're expectation values here, right? This was always expectation value.

So, what we're discovering is that the expectation value of the momentum is equal to mass times expectation of velocity plus Q times expectation value of the magnetic vector potential at the location of the particle. And there's a problem on the problem sets that tries to convince that this is the momentum of the Emag field.

The point is that if you move a charged particle, you are moving its electrostatic field. The electrostatic field causes the magnetic – the combination of an electric field and a magnetic field endows the – makes for a momentum flux in the now electromagnetic field, and there's a calculation which makes it look as if – I think it is probably broadly true – that the moving charge – to get a charge moving, you have give it momentum, but you have to give the field some momentum so this really is the total momentum of the things. It's not – the field – the particle is not on its own; it's not the only repository of momentum, the electromagnetic field is also a repository of momentum.

Anyway, so we have this non-trivial relationship now between momentum and velocity and again this is not something special to quantum mechanics; we've derived this in quantum mechanics, but it's a known result in Hamiltonian mechanics. Those people that have done F seven may have encountered this formula. I'm not whether it goes quite that far.

Let's have a look at the other equation of motion. We should have a look – which is harder; DBDT of the expectation value of PI is going to be of psi PI comma H – but what's H? It's PK – I'm going to write it out now – PK minus Q AK squared, summed over K commutator – stick in another psi. So this is the commutator of the momentum with the Hamiltonian where I have now written out the Hamiltonian in reasonably gory detail. Over two M. Over two M. I'm missing a one over two M, am I not? One over two M. In order that that's – so one over two M is the Hamiltonian.

Why does PI not commute with this? Well, obviously PI commutes with PK, that's not a problem. PI doesn't commute with this because sitting inside – because this is a function of X. So, when we work this out, we get a one over two M; psi, big bracket. We're going to have this thing commuting with that, so we'll have a minus Q PI comma AK. This is going to be summed over K. Maybe it would be better if we put a sum over K here. We've got to have one somewhere. Times PK minus Q AK. So that's P commuting with the first of these two brackets. And then we will have PK minus Q AK of PI AK commutator and a factor of minus Q from here. Close big brackets, stick in another psi.

So, this is the disgusting mess that we have. And now we have to address the question of, so what is this commutator? What is PI comma AK? We need it in two slots. We need it here, we need it here. Well, we've now used our rule for doing the commutator of a function of X. We used previously the – almost a rule for a function of P. The rule was that this is equal to D AK by DXI times the commutator, PI comma XI. The reason this is a function of XI, that's why this commutator fails to vanish. And we derived this rule quite early on that you can – by a tail of series of expanding your function – you can convince yourself that this is true that we just have a derivative times the commutator with respect to whatever it is we've taken the derivative in respect to. This is minus IH bar, so IDXI.

So, we're going to get some IH bars which we can cancel over there so, we're going to have DBDT of PI – the rate of change of momentum, which should be equal to force, all being well. This is turning out to be – yes, sorry. Yes. So, we're going to have a one over two M – no, we can take out a factor of Q over two M – this minus sign is going to cancel that minus sign. This Q I've taken outside. The sum over K has not collapsed; no, it's all there. Yes.

So, we're going to have psi as sum over K what of – we're going to have a DA K by DX I for this one here – hope we've got all the factors – PK minus Q AK, and we're going to have essentially the same thing but in the reverse order; PK minus Q AK of DAK by DX I. Close the big bracket and stick a matching `[[kep 0:34:06]]` psi on the outside to take away the expectation value.

So, unfortunately I cannot combine these two terms as they stand into one term because this is a function of X, which refuses to commute with that P. Similarly this one. So this thing is trapped on the left side of the P and that one trapped on the right side of P, and I can't combine it.

And in quantum mechanics, this is as far as I can go. I now have to – so, this is a respectable, totally above board quantum mechanical calculation. To go further I have to say, "Well, look; what am I trying to do?" I'm trying to recover the Lorentz force for you. I'm trying to show the classical – this is predicting the correct classical physics. If I'm predicting the correct classical physics, I can – if I'm talking about the classical physics, each of these operators can get replaced by its own expectation value. So the issue here is that here I have to take the expectation value of a product of one operator on another operator.

And such an expectation value is not automatically the same as the product of the expectation value of this on the expectation value of that. Because fluctuations in this operator maybe correlated with fluctuations in that. Sort of quantum fluctuations. But if we're in the classical limit, we don't worry about these fluctuations; we assume they all average away to zero, like the interference pattern associated with the bullets – I mean, the fluctuations average away, we're just left with a mean value, the mean. So we now replace this product with a product of - the expectation of a product – with the expectation values and then these become mere numbers – so this becomes an expectation value of this operator, the expectation of the value of this number – and then of course these numbers can be arrange in either order and I can stop fussing about this.

So we now say in the classical limit – we're specialising now to the classical limit – well, we can neglect fluctuations. We can write this as Q over M, because I'm going to combine these two terms, the sum over K of the expectation value DK AK by the XI times the expectation value of PK minus Q AK.

Now we can simplify again because this expectation of PK minus Q AK. Remember we showed above – we honestly showed above – without any fudging – was equal to the mass times the expectation value of the velocity where that's – yes. Alright?

So, this thing here can be replaced by M VK. And the Ms cancel. While we're about it, why don't we replace this PI with M – well, what we get from up there. I've lost it. Here we go. It's M VI minus – sorry, plus – QA. So this now comes down to DBDT of M VI plus Q expectation of AI. That's using that respectable formula up there for the relationship between velocity and momentum. Yes, that's correct. And that is going to be Q because the Ms are going to cancel the sum over K of DAK by DXI times – what did we say it was? – VK. And we ought to put an expectation value around everything because we are dealing now with expectation values we've explicitly gone to the classical regime.

Okay. So, we're nearly there. What are we trying to get? I'm trying to get that mass times acceleration is equal to V cross B. And it may look as if I'm still some way from that, but it's not so bad actually.

What we have on the left here is the rate of change of obviously the velocity and the vector potential evaluated at the location of the particle; not just anywhere else, but at the location of the particle. So, suppose we have a static field; static B field so that means that the partial derivative of A with respect to time can be taken to vanish. If this thing were non zero, it would generate an electric field. That you know right over time varying magnetic field creates – by Faraday's Law – creates a curly E field and that leads to a more complicated equation of motion. So, we're interested in static B fields so that the rate of change of the magnetic vector potential at any given point is zero, but this time derivative is not zero because the particle is moving and sensing the vector potential at different locations.

So what we have is that D by DT of AI is equal to – what's it equal to? It's equal to the – by the chain rule – it's equal to DXK by DT. Sorry, that should be a total derivative. DXK by DT by DAI by DXK. So, the reason that this quantity is changing is because the place where we're making the measurements is changing at this rate, and this is the rate at which A changes with location.

So, what we now have is that M D VI by DT – mass times acceleration – is equal to Q –there's going to be a factor Q on the right – I write down the terms I've already got – sum over K DAK by DXI VK. And then I'm transferring from the left side this times Q, right? So, it's going to become minus – this is VK and this is DAI by DXK. Strictly speaking, this bracket should be here because that summation sign is over both these `[[psi's 0:41:14]]` here.

Now this is actually equal to Q V cross B I'th component – well – because it's V across. Ask yourself what this would – I claim that this is true. Let us see whether it is true. In order to expand this vector triple product, we would say it's this thing dotted with this thing in the direction of that thing.

So if I expand this, I get Q. This thing dotted with this thing that means a sum over K of VK – VK AK. Direction of this thing gives me a nabla I. Because I'm trying to calculate the I's component. And then minus this thing dotted with this thing, direction of that thing. So that's VK, nabla K, AI.

It's a little bit of a complicated vector triple product because this is a differential operator and it is operating only on this and that's why I've written them in that form. It's this thing dotted to this thing, direction of that thing but this is only working on that and then it's this thing dotted with this thing direction of that thing. That's nice and easy.

And I think you can see that this term is this term and this term is this term, if you move that around and back, right? These two terms are the same.

So we have indeed recovered mass times acceleration is equal to Lorentz force. In the classical limit.

Well, I think that that's really all I want to do. Yes, that's all I want to do. That justifies provisionally the use – so this Hamiltonian – where was it? – P minus QA all squared of two M being - that change in the Hamiltonian introduces a magnetic field into the physics. And we will use that when discussing atoms down the track. And if you look at the back end of chapter three, you can see there are some quite entertaining things you can do with the motion of a particle in a uniform magnetic field when it turns out that you can recycle the physics – no, you can recycle the formulas and the mathematics that we did for the harmonic oscillator. You can recycle it for this uniform magnetic field case.

But the basic principle is that if you have a uniform V field and a non-relativistic particle moving in a uniform – charged particle moving in a uniform V field – you can have the orbits as circle – the particle circles around in this V field – with some radius that depends on its speed. If you have a fast particle that goes round 2- we have MV squared over R is equal to Q VB. So we have that V over R is equal to QB over M is equal to the Larmor frequency, so the angular frequency, on which the particle goes on its orbit, depends on the strength of the magnetic field and the charge of the mass but not on the energy; it doesn't depend on how fast you're going.

So, fast particles go on big circles and take the same time to go around the slow particles. So you have a characteristic – all the motion is that some characteristic frequency – and that is reminiscent of a harmonic oscillator and allows us to – that's the fundamental underlying physical reason why we can solve the problem of motion, the quantum mechanical problem of motion in a uniform magnetic field using the apparatus of the harmonic oscillator.

So I think you should have – I mean, I hope some people will have some fun looking at that in the vacation, It is very good quantum mechanics. It's very important physics, but unfortunately, we are not going to have to cover it in the lectures. But the magnetic field will be important in the context of atomic physics.

Okay. So that's it until next time. (Applause)