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Contributor So yesterday we looked at this, this pair of wells that were separated by a barrier so that classically the particle couldn't get from one well to the other and we found that the particle got from one well to the other and used that to make a model of an ammonia maser with the nitrogen atom passing through the barrier formed by the hydrogen atoms.

So let's look at this phenomenon from another perspective, the perspective of a scattering experiment. And we'll come on to what this has to do with radioactivity I hope at the end. So consider this set up we have a potential barrier here of height V0 usual square form for computational convenience. And we have some incoming stream of particles. We have a beam of particles here represented by the wave function A+eikx. So these are of course particles with well defined momentum approaching the barrier.

We expect to see some of these particles reflected classically if the energy of these incoming particles were less than V0 they would all be reflected so we put in a reflected wave here which goes like E to the minus ikx. Remember the time dependence of everything in quantum mechanics these are – for a state of well defined energy is E to the minus IE upon H bar T. So these things are going, having sort of E to the minus I Omega T type dependence and therefore if you have a minus sign here you're looking at a wave which is travelling to the left, if you have a plus sign here you're looking at a wave which is travelling to the right. And then we expect some of the particles to get through so we put in a wave in this portion we say the trial solution should look like C with some unknown constant C E to the ikx. And then within the barrier if – because we're going to look for solutions at energies lower than V0 we're going to have B+E to the KX plus the trial solution will be a combination of an exponential growth and an exponential decay.

So this is a more complicated, so this is a bit different from what we've done before in two ways. One is that – well the main thing is that we are initial – our problem inherently has a lack of left right symmetry right. We, the potential that we're discussing here has left/right symmetry it's symmetrical around the origin which I forgot to say but the origin is here in the middle of the well, this is minus A and this is A over here.

So the potential is going to be an even function of X same as ever but our problem, our initial conditions the physical situation we wish to discuss has a built in asymmetry because the particles have to come in from one side or the other. Now we could – so that's... And that is computationally very inconvenient. It stops us using this nice trick we'll it doesn't – it makes it difficult for us to use this nice trick of looking for solutions to the problem which have well defined parity and thus discussing only what happens at this boundary condition.

With set up we're going to have discuss this boundary condition and this boundary condition because you can see there are fundamental differences between what's happening on those two sides. Now you can handle this problem using looking for solutions of well defined parity but it's slightly unnatural and I think – well it's actually a very good way to go but it's not such an obvious and intuitive way to go even though it's computationally simpler and I think it's worthwhile just seeing what happens when you play the game straight forwardly and you'll see the algebra comes quite unpleasant which illustrates the benefits that we had before by assuming well defined parity.

Alright the other thing that's different here is that because we are considering particles which are free, you know, that – because the potential goes to zero outside this interval here the particles – we're going to consider particles with positive energy, the particles are going to be able to push off to infinity so we're not going to find discrete energy levels. We're going to be able to find solutions for any energy, right. That's... whereas previously we had a potential which `[[?? 0:04:42]]` going to infinity and that made the energy levels discrete.

So those are the differences because we're dealing with a different physical situation and it has implications for the maths. Right so what do we have to do well it's very boring we have to impose continuity of the wave function. So the wave function here is this wave plus this wave and that has to be continuous at this boundary X equals A so it has to give you the same numerical value as the sum of these two things. So let's just quickly write that down so we have A+E to the minus iKA which is the incoming wave evaluated at that barrier. X equals minus A plus A minus E to the plus iKA and that had better equal B plus E to the minus KA minus, sorry plus B minus, E to the minus, no, no that one's got plus KA.

Right, many double negatives here unfortunately. Oh we forgot, I forgot to say of course that we'll have as ever that K is equal to the square root of 2M times the energy over H bar squared because P squared over 2M energy and P is H bar squared K squared, sorry P is H bar K. And we will have that big K is equal to the square root of 2M V0 minus E over H bar squared.

So this is the condition for the wave function to be continuous at X equals minus A. We require as yesterday that the gradient of the wave function is also continuous there so we have to take the gradient of that function on that left and valuated X equals minus A and we find that IK, common factor – A plus E to the minus IKA minus A minus, E to the IKA close brackets is equal to big K common factor B plus E to the minus KA, minus B minus, E to the KA close brackets.

Then we have, so it's two equations. Now we have two more equations because we have to get everything hunkey dory on the right hand boundary which is not now dealt with by symmetry as it was yesterday so this is the – this is where life becomes – everything becomes difficult. So we have C E to the iKA is equal to B plus E to the KA plus B minus, E to the minus KA and we have that iK over K – I'll write it thus of C E to the – or may be I should do that one there. C E to the iKA that's the gradient on the right side is equal to big K common factor open brackets B plus E to the KA minus B minus, E to the minus KA, close bracket and I live in hope and some anxiety that that has been – those equations have been correctly stated.

So what do we have to do? We now have four equations and five unknowns I think, right. There are two As, two Bs and a C. So we will not be able to get rid of all of them. We will be able to express in principal any one of A B – of the A Bs and Cs in terms of the other one and that physically corresponds to the point that the flux of incoming particles is controlled by A plus. And that's in your control. You can put in more particles or fewer particles and that will obviously lead to more particles coming out or fewer particles coming out depending on the incoming flux.

So the general idea is you expect to be able – the goal is to express any one of these things as a function of A+, as a multiple of A+ and we expect them to be linear in A+. So that's why we've got two few equations. We don't physically expect to be able to determine everything. So what we should do is engage in an elimination exercise, a reasonable way to go is to take these two equations here divide this equation by this equation say and that will get rid of C and will give you a relationship between B plus and B minus.

And then you can take that relationship between B plus and B minus and use it in these two equations to express the right – to get rid of B minus from these equations – these two right hand sides so they both become simple multiples of B plus. And then you can divide these two equations one by another, the B plus which would be a common factor on the right hand side will go away and you will be left with a relationship between A plus and A minus. The single relationship between A plus and A minus so that will be the promised relationship that expresses the number of reflected particles as a multiple of the number of incident particles.

So once you found what A minus is in terms of A plus you can go back to your original expression here which had only B plus on the right hand side – A minus can be expressed as a function of A plus and you can well define what B plus is, well define what B minus is and they can all be determined.

So let me not do all that algebra. That's the strategy, the execution of course is quite tedious and the scope for making errors is quite large and in fact I find that there's a typo right there in Equation 540 in the book because when you do eliminate between these two equations here to find out the relationship between B minus and B plus it should be that B minus is 1 minus IK over K over 1 plus IK over KE to the 2 big KA B plus. So that differs from what's in the book partly by arrangement of this but more importantly by this having been left out that's got – that's slipped out in the – during the typesetting.

Okay so we have that relationship there. We stuff this back in to the other places and we find that A minus is equal to A plus so I've described how we – what we do. We take this B minus use it to get rid of B minus from here and replace that with B plus and some factor then we divide these two equations and then we get this relationship I'm about to write down between A minus and A plus and it is A minus is A plus E to the minus 2 IKA Q minus 1 over Q plus one where Q is itself pretty yucky it's Cosh 2KA minus I K on K, then hyperbolic Sinh of 2KA all over Cosh 2KA minus big K over IK of Sinh.

So the algebra is as promised altogether more messy than it was yesterday because we're not exploiting parity, we're not dealing with finding a…

Right so what do we want to know about this physically. What we want to know about this physically I think is what is the chance that the particle is reflected, what is the chance that the particle gets through. So classically everything will be reflected and the modulus of A minus would be the same as the modulus of A plus, right. And you can see that that isn't looking very promising because that would require that well basically the Q was simply enormous right. If Q were very large then Q minus 1 would be the same as Q plus 1 and everything would be reflected. But in reality it's not all going to be reflected some thing's going to get through. How to find C? Well we could, you could take this A minus expression from it as I've described obtain B plus from B plus obtain B minus to put these back into this equation here say and find C. That's too much like hard work.

It's easier to say that look there's going to be conservation of particles. We've got a well defined theoretical apparatus here which is not going to – which conserves probability. So the incoming particles the A plus are all going to go out at the end of the day either to the left or to the right. So we can argue that A plus mod squared which is – well that is the spatial density of incoming particles if you like. If you multiple that by the speed of the incoming particles which is P over M so H bar K over M you will get the flux of incoming particles and the flux of incoming particles has to equal the flux of the outgoing particles which is A minus – the mod square of A minus, the density of outgoing particles again times H bar K over M for the speed plus C mod squared, right.

So conservation of particles implies this relationship between these amplitudes. And of course you can in principle check whether this algebraic relationship is satisfied by these equations by hard slog because I've described how you can in fact find C. We've already found A minus you could in fact in principle find C and check that it's satisfied this equation but we don't want to do all that algebra.

So the point is that what we wanted to say is the flux – well what we want to say is the following actually it's the fraction of particles that get through is obviously the ratio of the incoming flux and – well the ratio of the outgoing flux to the incoming flux. So it's going to be this fraction that we want we'll call it F is going to be Mod C squared over A plus squared. Because the constance of proportionality namely H bar K over M between this quantity and the outgoing flux on the right is the same as the constance proportionality between this constant and the incoming flux on the left.

So the fraction of particles that get through will be given by this ratio here which given that relationship – so in other words C well let's – we can write that now as A plus mod squared minus A minus mod squared of A plus mod squared. But we've got A minus mod squared from that expression at the top and as a multiple of A plus mod squared so we can write this as 1 minus Q minus 1 over Q plus 1 mod squared.

Well the mod square of this ratio is the mod square – the ratio of the mod squares of the top and the bottom so this can be written as 1 minus 1, Q minus 1 mod square over Q plus 1 mod square. So let's address ourselves to what these mod squares are. So what's Q minus 1 well Q minus 1 is going to be well it'll obviously – on the top it will have the existing top minus the bottom. So when we take away the bottom from the top the Cosh's go away and we are left with I think K over IK minus IK over K times Sinh 2KA. And that will be over – I'll just call it the bottom because it's the… we're not really going to take much interest in what this bottom is, it is the bottom that you see up there Cosh 2KA minus K over IK etc. And Q plus 1 – the reason we won't care about the bottom is of course it will cancel when we take this ratio.

So for Q plus 1 we unfortunately find that the Coshs add and the Sinhs irritatingly refuse to cancel so this becomes 2 Cosh 2 KA we're adding so we have minus IK over K plus K over IK all in a bracket Sinh 2KA. And again that's over the bottom.

So what we need to do now is take the mod square of these two numbers, ratio them and take it from 1. So the fraction that gets through is going to be 1 minus – so the top of that is completely imaginary right, it's pure imaginary. We should take out an I from that bracket and then we will find we're staring at K over K plus K over K squared times Sinh squared 2KA. So that's Q minus 1 mod squared as regards the top the bottom we're not interested in because then it's cancelled with the other bottom. And now we have to put underneath the mod square of this which will be 4 Cosh squared 2KA right because this is the real part of it. This is the imaginary part of it. We take out a Factor I and now we're staring at plus K over K minus K over K squared Sinh squared 2 KA. Nearly there.

So now we put this all – these two bits. It'll simplify if we put these two bits on a common dominator so the top one is on a common dominator will be this bottom plus that stuff there. So this will be 4 Cosh squared 2KA. And now we're going to have Sinh squared, let's write it in, plus – yes Sinh squared 2KA brackets – now brackets what. We will have this brackets squared well we'll have this bracket squared sorry minus this bracket squared. And when we square these brackets we're going to get K squareds over K squareds which will cancel because of that minus sign. And what will not go away is the mixed term, the product of multiplying this on this which generates 2 and the product of multiplying this on this which generates another 2 so we will get 4. And it will in fact be with a minus sign because this minus sign is there, you know, when this comes up here that minus sign will stick out and this minus sign will make the mixed term minus there.

So this is going to be times minus 4 and it's over the bottom as you see it over 4 Cosh squared 2KA plus K over K minus K over K squared Sinh squared 2KA. And the top simplifies must beautifully because Cosh squared minus Sinh squared is one so the 4s can be cancelled and this actually is nothing but 1 over – so the fraction is 1 over Cosh squared 2KA plus a quarter of K over K minus K over K squared Sinh squared 2KA. Not much fun.

So what do we learn from this. What we learn from this is most interestingly is what happens if we have rather a high barrier and the particles are very short of energy to get through, alright. So K is a measure of the deficit in energy that the particles, right that the – by how much they don't have enough energy classically to get through the barrier. If the barrier's very high and they don't have much energy then we're looking at the Cosh of a largish number and the Sinh of a largish number and so what we can say is that for large KA we can say that Cosh 2KA behaves pretty much like Sinh 2KA behaves like E to the 2KA. Alright?

But we're interested in fact in Cosh squared and Sinh squared so F is looking like 1 over E to the 4KA. So if KA is an appreciable number this probability of penetration is becoming small, crucial result is that the probability of getting through there is decreasing expedentially fast in the height of the barrier. So you don't need a very high barrier to make this quite a small effect. And somewhere here we have… So this is – ooh hoo – this machine goes to sleep doesn't it that's the trouble. It shouldn't go to sleep. Give up – is there anything there. I'll just draw it.

Is it sort of? Yes. No doubt we're saving the planet by having the machine turn itself off but yes okay. I can't see it. So what – you want to do some so that's sort of asymptotically what happens when KA's very large. In detail you might want to know the smaller KA – sorry.

Right so these results are for a barrier which is so in these results the barrier is not terribly high. So we have V0, sorry we have that E is equal to 0.7 V0. No, no, no, sorry what have I done? What have I done, what have I done with that? Yes that is correct, sorry. Yes the height of the barrier is, sorry there's this parameter W isn't there which we talked about yesterday which is a measure of the width and the height of the parameter, of the barrier. So it's 2M v V0A squared over H bar, this animal, right. That's your dimensionalist measure of the height and the width of the barrier in terms of the mass, the particle with no reference to the energy of the particle.

Sorry, that's not the case. Then what's being plotted here is the probability of getting through as a function of energy over V0 for barriers of different Ws. So I think is it Point 5 at the top there? Yes. So here's a relatively weak barrier which gives you fairly small energies a chance of getting through in other words it's not a very fat barrier which is the crucial thing. This is a fatter barrier, this is a fatter barrier and so you can see how as the function of the energy your chance of getting through rises in detail. Okay.

See if we can get this thing to stay alive for later. What's physically interesting about this or an interesting application of this is to radioactive decay. So this is obviously a very simple minded model that we have so far but the general idea for example is this. So what we should say is that inside 238 Uranium which is the non fissile sort of uranium you have a number of alpha particles. This is a simple minded picture so what does the potential energy of an alpha particle – so we kind of consider this to be so 238 Uranium which decays to 234 Thorium and alpha particle with a half life of I think it's 6.4 giga years. Right, so it takes the age of the universe typically for a Uranium 238 made in some supernova to eject an alpha particle.

So what's happening here from this perspective. What's happening – so what we should do is we should think about this alpha particle and this Uranium 234 nucleus as a kind of dynamical system. So the alpha particle when it's a long way from – when it's a decent distance more than 10 to minus 15 metres or so away from the Thorium nucleus is repelled by the electrostatic repulsion so the potential energy curve has a sort of 1 over R type behaviour here. If you get – when it gets close enough to the Thorium nucleus, the strong interaction and it's able to exchange `[[Gloynes 0:29:40]]` and stuff with the alpha particles, well with the nucleons inside there and it feels an attraction so there is a well that looks a bit like this except this is extremely narrow. So the width of this right is say 10 to the minus 15 metres, a sort of typical nucleus size.

So inside that Uranium 238 that you mine in Australia or something there's some alpha particle moving around in here with a large velocity a sort of relativalistic velocity motion inside nucleae is kind of relativistic. So it bangs to and fro across here right if you're moving – if you've got 10 to the minus 15 metres to cover and you're travelling at some speed comparable to the speed of light that means that you cross this thing – what does this give me 10 to the minus 23, yes. You need about 10 to the minus 23 seconds to cross. So roughly 10 to the 23 times a second this alpha particle bangs to and fro, to and fro, to and fro. This will be the classical picture and it needs to do this. So it does this for on the order of 6.4 giga years so for many giga years. So for on the order of shall we say 10 to the 17 seconds which is a third of the age of the universe so it makes about 10 to the 40 impacts on the barrier. And then wonderful moment it gets out on the 10 to the 40th attack, whatever. It slips through here and goes off to infinity.

So this astonishing phenomenon of a systems with incredibly small dynamical times, the smallest dynamical times, you know, in the typical physical world doing something on a timescale which is the age of the universe it is the most astonishing phenomenon but how does it happen, it happens through this exponential decay, the height and width of this barrier are substantial but that E to the 4, it's that E to the 4 times the height and width of the barrier amplifies this so much that your chance of getting out turns out to be only 1 in 10 to the 40 so that a neutron that got trapped in there in a supernova before the son was born pops out today.

So we should now – so that's the end of games with square potential wells. I hope you get the idea that it's a rather artificial it's a scheme for finding solutions to the time independent Schrodinger equation which can illustrate interesting physical phenomena although it's the potentials themselves are very artificial and we should now just ask ourselves what of the results that we've obtained would be spoilt will change if the changes in potential weren't abrupt. Right, so in the real world they're not going to be just step potentials. We've just stepped potentials as a computational convenience. In the real world they're going to have to extend over some distance and one wants to understand – it's important to understand which of these results would survive and which would be spoilt by taking a more realistic potential. And I've focussed on problems where stuff would survive and tried to neglect problems or having spoken about problems which would be seriously damaged but you can be misled.

So in particular, if you if we would do a calculation precisely analogous to this for particles encountering a square potential well we could – all this calculation could be pushed through with the minor modification that in here we would have B plus E to the I KX and B minus E to the minus I K big KX. Right, we would have two. So if we had particles moving in here from infinity with an energy greater than zero the particles when they got here would speed up and slow down when they got here and stuff. And classically all the particles would pass through.

If you solve this problem using this apparatus here what you're going to find is that some of the particles are reflected from this barrier, well some of the particles are reflected sorry from the whole set up – I don't want to say which barrier they were reflected from because there were two barriers they can be reflected from and the results are a super position of those. And some particles get through.

And if you do this calculation you were learning something which will be profoundly changed if you're more realistic and say well my real potential well of course is going to have somewhat slopey, you know, some what slopey boundaries. And the issue is how steep does something have to be for this to be a decent guide. The good news is that the results to that kind of calculation are not going to be profoundly effected if by the steepness. They'll be somewhat affected but not enormously affected. So long as we stick. We would be misled if we put particles in at sufficient energy that they were classically able to get over the top but if we stick to particles which are classically forbidden in here we're not going to be enormously deceived by taking sharp boundaries.

How do we do this? Well what you need to do is numerically solve the time – solve the wave equation - solve the time independent Schrodinger equation for some kind of a potential change which can be made either steep or less steep. So if you take the potential as a function of X is equal to sum constant brackets times 0 if mod – if X is less than minus A. And in this zone here is something like 1 minus, sorry 1 plus sine pi X over A. That's for mod X less than A and you take it a one down here if Mod X if X is greater than A. I hope I've done that the way I should've done that.

Then you will – so this is just a simple functional form that describes a curve that looks like this. Right it goes from V0here it's precisely V0when you're more than A away. And it's precisely 0 if you're the left of minus A and it moves smoothly and continuously with the continuous gradient from one thing to the other thing. And by changing A you can make this steeper or less steep. And it's very straight forward I urge you to try it on your laptop to solve the time independent Schrodinger equation numerically. There's a problem describing how to do I think it's certainly in the book, possibly in the problem set. And what do you find when you do it you get this kind of curve here. So this is the reflection probability as a function of KA so – and that's right. And this is for – this is what I did for an energy E which was equal to 0.7 V0. So all of these solutions are for energy E equals point – is point 7 V0 which in the square with the – if we have an abrupt, you know a sudden change in the potential gives us this probability of roughly point 1 of being reflected.

Sorry is that the... This is the property reflection. Did I say something different? This is the probability of reflection and the square one gives you, the sharp one gives you this. The numerics reproduce this if you take KA and A is now this – not the width of a well but the width of the transition or 2A really is the width of the transition. If KA's less than 1 then the numerics reproduce the analytic solution but if KA's bigger than 1 you see there's a very – look at this it's a logmirthric scale right this is a probability of .1, .01, .001. So the probability of reflection drops like a stone as KA becomes bigger than 1.

So the abrupt transition is going to be profoundly misleading when – unless the transition. So the step – in this case where we have – what's crucial here is that we have a transition from between two zones within which the particle is classically allowed. Right, so the step between classically allowed regions is misleading – it exaggerates reflection if KA is greater than on the order of 1.

That is to say – what does that tell me K is 2 pi over lambda so that tells me that A, if A the transition width is greater than 2 pi over the `[[Debroy 0:39:51]]` wavelength, right. So the transition really has to be quite abrupt in terms of this natural sense of scale. If you ask – so what's the Debroy wavelength for an electron the answer is that it is on the order of 1.2 time 10 to the minus 9 energy over 1 EV to the half metres.

So the Debroy wavelength this quantity for an electron and because I've mentioned electrons obviously because they're things that we do fire around our laboratories. People used to fire them all around their homes even when they had cathode ray tubes. So it's a typical kind of particle you want to understand about.

Then the Debroy wavelength is a nanometre or so times the energy in electron vaults – oops that's a minus a half isn't it because the higher the energy the shorter the Debroy wavelength.

So if you were constructing a step potential typically you're going to be doing it by having some kind of – doing some kind of solid state physics so that those sheets of glass provide pretty much a step change in – they provide a change in the refractive index which affects photons, right. So a photon sitting in the window have a chance of being reflected, a chance of being transmitted basically as if it were being bounced off a step potential, why because the photons have wavelengths – those photons that we're bouncing off the windows have wavelengths of 500 nanometres or something and atoms. So the size of an atom is of course on the order of .1 nanometres.

So it's easy using atoms to make changes that occur over a few atoms therefore over – on the order of a nanometre. You – so you can make if you are using atoms to make the barrier, you know, you're propagating your electrons through some kind of solid state material you can probably make a step change which has a – you can change the effected potential the electron experiences within on the order of a Nanometre. So you may be able to get useful results out of this provided your energies are lower than 1EV but that's extremely challenging. In practice your energies will typically be higher than 1EV. So these results are going to be basically misleading.

What do you see here is return of common sense and rationality. If you roll a piece of chalk, you know, off the edge of this table it will of course fall. It won't be reflected. It's not going to be reflected by the lower potential, the on set of lower potential. And that's what the numerics are saying here that unless you have a – that in practice when something encounters a drop in potential for example the reflection chance is going to be in fact very small because this is not going to be abrupt it's going to be like this. A tiny bit easy and then everything is basically going to get through.

So what happens, what actually happens is that when you have a slow change, a gradual change in the potential is that the wavelength – as the electron or the particle comes along it comes to this region of lower potential energy. We would say it speeds up. The numerics will show you that the wavelength of the wave is getting shorter so the momentum is getting larger because as P is H bar K. Yes it's speeding up and it's just – there's no reflected wave so the whole thing just moves in some new regime with a shorter wavelength of everything changing continuously.

Well I think that's pretty much all I want to say so we'll finish there and that's the end of step potentials and on Monday we can start on…
