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Contributor Okay I guess we should – we should get going. So we have on – we have today to discuss some unfortunately rather formal stuff and tomorrow we will do something that's physically more interesting, the Einstein Podolsky Rosen experiment but which will draw heavily on what we're going to do today. So what we want to do today is face up to the fact that many of the systems that we want to apply quantum mechanics to come in parts. So for example the hydrogen atom consists of an electron and a proton and to know the state of the hydrogen atom you want to know the state of the electron and you want to know the state of the proton both.

A diamond consists of on the order of – so condensed metaphysics is about things like diamonds which a diamond would contain 10 to the 23 or whatever carbon atoms. And to know the state of the diamond officially you would need to know the state of the 10 to the 23 carbon atoms. So it's going to be important to move forward towards applying quantum mechanics to any non trivial really interesting system we're going to have to learn how to describe systems that come in parts. And this turns out to be – quantum mechanics has its own way of doing this which is actually very elegant and powerful but leads to some surprising results.

The hydrogen, in a hydrogen atom the electron of course is strongly interacting with the proton. It is electro statically attracted towards the proton and in a carbon atom – sorry in a diamond the carbon atoms are obviously very tightly coupled to each other by covalent bonds or whatever. So there's a – there are springs as it were. There are things connecting the different parts. But it turns out that the quantum mechanics of a system made up of two objects is non trivially, is non trivially different from the quantum mechanics of the two isolated things even if you just logically consider them to be the same. So when we do angular momentum we will – in the coming weeks – we will find that very strange and interesting results arise just because we put two gyroscopes in a box with no physical connection between the two of them and start asking questions about what's the angular momentum of the box as opposed to what is the case of the individual gyros?

So knowing the state of the well defined states of the box turns out to be very different from knowing well defined states of the individual gyros. So there's a – what we're talking about today is putting things logically together to make compound systems and there may or may not be springs connecting, physically connecting these things. Alright…

So the central problem of the – the central thing we have to address is if we have a system A and a system B. So we have two distinct systems and this one lets say has states IA. Alright, so this indicates which system we're talking about then there's supposed to be a semi colon here and there's an index here which tells us which of this system states we're addressing. And we have a system B and it will have states something like this. And what we want to know is so how are we supposed to write the states of the compound system, the system that you get by considering A and B together. So this might be the electron, this might be the proton what's the state of the compound system which we call a hydrogen atom for example?

Alright, because if we know how to add – if we know how to compound a, system A with system B to make a combined system we can compound another one – we can use the same rule to add another element of a bigger system A, B, C and so on and so forth. And we can do – eventually we can build up a diamond of 10 to the 23 carbon atoms. The central, once you know how to add two systems by repeating this process adding more and more systems you can put any number of systems together. So this is the central problem that we have to address.

So the states – what we – first of all we just write some formal stuff: the states of the compound system AB. This is when you logically think of the system at A with B as one system are of – well some of the states of the system AB are written like this AB semi colon I J and we write it symbolically as A semi colon – whoops – I B semi colon J. Sorry the semi colon and the J look too similar.

So what is this? Let's just ask ourselves what does this mean? This means the state of the compound system where the state A – where the subsystem A is in its 'I' state and the subsystem B is in its J state. Right, so this is – we know, we have to know what this means and I think we do know what this means I've just given words that say – give meaning to that. And on the right hand side we have a symbolic multiplication and we don't need to worry too much you'll see as we go on that we don't need to worry too much about what exactly we mean by this multiplication. But this is just a symbolic product of – efficiently it's tenser product but we don't want to frighten everybody. This is a symbolic multiplication of a ket on a ket – alright? We'll find out how to interpret that as we go along.

Then we will obviously we can have the bras – there must be associated bras since this is a state of a system it has a bra which will be I J – whoops – is equal to of course the logical product of the bras. And we give meaning to this thing by explaining what happens when this goes on to this. So when this goes on to this we should get a complex number so I – to give meaning to all this I need to explain what this is I primed J primed on AB AB semi colon I J is equal to. Right, so we need to give – this should be a complex number to give meaning to all this real-, hocus pocus I need to explain which complex number.

The complex number it is this complex number AI primed AI times BJ primed BJ. So what I've written on the right makes perfectly, is completely well defined because this is a complex number and this is a complex number and we can multiply complex numbers. And we get a complex number which gives meaning to this on the left. And that's really, really, really the – all we… That's the essence of giving meaning to this thing here because remember we only want these kets all we want with kets is in order to calculate amplitudes which – whose mod squares are going to be the probabilities for some – give us our predictions. So as long as you know how to get amplitude out of a ket you know enough about the ket to get on with it. Right?

So we've given meaning to the process by which we extract amplitudes out of kets which is this brahing through business, because that leads to the experimental predictions which are the whole point of the theory. Okay why is this – why does this make sense? Why is this, is a sensible definition, right. So this is a definition of what we mean by these animals. Why is it a sensible definition? Well it says that the probability of getting – of measuring the results I primed and J primed given that we're in this state is equal to obviously the mod square of this horrible thing ABI primed J primed ABIJ mod square, right. That's how we would interpret this complex number. And according to this formula this is equal to the product of the probability associated – this is the probability for the system AB – the probability of system A of getting the result I primed times the probability of getting the result J primed.

Right, because if I take the mod square of both sides, the mod square of this product is the product of the mod squares. The mod squares on this side by definition are these probabilities that… So the probability – this says the probability that if I'm – take measurements on my combined system I find that A is in the I primed state and J is in the – B is in the J primed state is simply the product of the… It should be the product of the probabilities that the A system is in the I primed state and the B system is in the J primed stated if you make individual measurements.

So that makes perfect sense and it's motivating – this is why we write the ket of the compound system like this. This multiplication rule, this symbolic multiplication is inherited from this law for multiplying probabilities in probability theory, okay. Now that, having said that and everything's nice and simple we have to make the point that – and now I want to show not all states – this is the thing that's surprising of AB are of the form AB. That's what – I want now to establish that this is – that it's not true that all states of the system are of this form.

Okay so let's, so for example consider two state systems. So we're going to do a concrete example to illustrate this general and very fundamental principle. We're going to have two two state systems. We're going to have A who's going to have states plus and minus, these are a complete set. So we're considering the simplest non trivial example and B is going to have the states up and down, alright. This is just notation that enables us to – by using a plus sign and a minus sign for A and an up arrow and a down arrow for B I avoid the necessity of writing down these pesty AB labels, right.

Let's now consider. Let the state of A B A plus plus A minus minus – so this is a general state, okay. By taking a linear combination of the two bases vectors of my two bases states in my two state system I write down a general state by choosing these amplitudes to be whatever you like you can make any state of A whatsoever. And let the state of B similarly be B up up plus B down of down. Then what's the state – now let's have a look at the state AB, the state of AB that we get. Well it's going to be this thing bracketed into this thing A plus plus plus A minus minus. B plus plus plus B – oops sorry, sorry – this has the up and the down states. B subscript down ket down. And when you multiply this out you get a disgusting mess, right because you get A plus plus – sorry, sorry. A plus B up of plus up plus A plus B down of plus and down plus A minus B up of minus and up. Plus A minus B down of minus and down.

So my – so this state is now along, it's now a linear combination of four states and it's – this is strongly suggesting that these four states are bases states for the compound system and indeed we will show that they are times amplitudes which are these products of those individual amplitudes. And these amplitudes have well defined meanings, right. So for example A minus B plus is the amplitude that A will be found minus and B – what did I say? Up. So take the mod square of this you get the probability that the experiment to measure As property and Bs property will be these particular values.

But what I'm trying to show is that this state is not the most general state, okay. And the way I'm going to do that is I'm going to calculate the probability that B – may be you should do this the same way I've got it here. That B is up given that A is in the plus state, alright. So this is the kind of – so if – this is a reasonable question. We've measured and found that A is up and I now want to know okay so suppose I measure B's property will I – sorry I found that A is plus – will I find that B's property is up or down. This is going to be the probability that given that I am where I am B will be found to be up.

Okay well this is equal to the probability simply that we have up and the probability for being up and plus over the probability that A is plus. Now why is that? If I would move this here then this would say that the probability of being up and plus is the probability of being plus times the probability that we get up given that we have plus. This is a very important result from statistics. This is classical probability theory this is known as Bays Theorem. But it's really a trivial rearrangement of the rule for multiplying probabilities. The probability to be up and plus is the probability for being plus times the probability if you are plus that you're up. So this is nothing to do with quantum mechanics this is just a rule of probability theory which now plays a very important role in statistical inference in all the sciences physical and social.

Alright so what is that? That's the probability that we're up and plus over the probabl-, this is the – having plus on A comes in – we can have plus in A in two ways with a competent system. We can have it either with B down or B up and they're mutually exclusive events so I can add their probabilities. So this probability on the bottom is P up plus plus P down plus.

So what is this? This is equal to 1 over dividing through 1 plus P down plus over P up plus. What about this? Let's go back to that expression up there. What's this probability? What's this probability in terms of those amplitudes? P down plus, P down whoops – down and plus is equal to A plus B down. And P up plus going up there, P up plus is A plus B up. So these A pluses cancel – oh and we need to take the mod square of this whole of course, right. But these – the crucial thing is those things cancel so this is in fact equal to B down plus – sorry this is equal to B, this ratio.

So what's the point? The point is that this probability is actually – we've just shown it's independent of A plus and A minus. So this probability does not depend on the state of A. What does that mean physically, historically? It means that the systems are not correlated. I've just calculated one specific conditional probability but you could calculate any other conditional probability and you'd find the same thing. That the probability of any state of B is independent of what you assume about – what the result of measuring A and so on. These are uncorrelated systems.

So what we conclude from this is that when the state of AB is a product of a state of A times the state of B the systems are uncorrelated. And that's an important physical assumption. Now, for example, if you have a hydrogen atom is the location of the proton correlated with the location of the electron? Well of course it is because if the hydrogen atom is here you can be pretty damn certain the electron lies within a few nanometres or if you – within a manometer of the proton. If the proton's over here you can be pretty sure that the electron is within a nanometre of the proton. It's over here. The electron and the proton are very strongly correlated because they're – you know they're physics. There's a piece of Hamiltonian which is correlating them.

So we don't, yes. So we do expect systems to be correlated and that means we do not expect systems in general to have wave functions that look like – to have states that look like that. So let me see, the point is that the – I'm not going to go through the demonstration I think that I said – let's go back up some way. Let's go back to here. So if these objects form a complete set of states of A and these objects form a complete set of states for B then it's not hard to persuade yourself that – sorry that these objects form a complete set for AB, alright? So this is a complete set if these complete for their respective subsystems.

Now what's this telling us? This is telling us that any state of the system including correlated states which as I've tried to argue are natural states, states in which the two subsystems are correlated they must be writable as linear combinations of these objects. So the conclusion here is – let's put that back and start over here. So any state of AB can be written as AB equals the sum CIJ summed over I, summed over J of states A I B J. These states are described uncorrelated, states in which the two subsystems are uncorrelated. But this may be correlated, probably is correlated. So the way quantum mechanics introduces correlations between subsystems is by taking linear combinations of uncorrelated states.

We just had such a linear combination of uncorrelated states here, right. And it turned out that in this case that was still an uncorrelated state because this was simply an expansion in terms of some bases states of a state which was already a product of just two states. So the point is that the general state cannot be written. This thing in general cannot be written like that even though when you see a long list of bases states it may, you know, with certain complex numbers in front, it may be that the state can be written thus.

So whether this thing can be written as a product of two separate states depends on these numbers. Now we haven't got time to go in to what property it is of these numbers which ensures that you can do a decomposition like this into uncorrelated states, which makes this state uncorrelated. And when these are correlated but you can find a complete account of it in the book. There are I think some – and there are problems investigating this. But the point is that if you – in this concrete example here, right. This is one of the Cs, this is another of the Cs, another of the Cs, another of the Cs and these Cs are not general. They have the property, you could arrange those in a two by two array of objects and if you – this matrix of – this two by two matrix is a sort of a degenerate matrix. It's a special matrix it's not the general one that you get by making – choosing these numbers independently.

So correlations go in like that. And in quantum mechanics when you say that two states – two systems are correlated you actually usually use the word 'entangled'. Entangled is just the same thing it's just quantum mechanical jargon for correlated. And it – what it means is if a compound system or two subsystems are entangled it means the state of the compound system cannot be written in that form. It has to be written in this form and these numbers do not have the property that requires them that they have to have to enable them to be expressed as products of individual, amplitudes for the individual systems.

So let's do a little bit of quick counting. Suppose there are M bases states of A and N of B. Alright so there are M values that I can take and there N values that J can take. So then they'll be M times N amplitudes CIJ. So to specify a general state of the system you need to specify MN numbers – CIJ. To specify a state but to specify A you need just M numbers AI. And to specify B you need N amplitudes BJ so to specify a general state of the form AB you need M plus N amplitudes. So M plus N is generally much less than MN.

If we're two… In this little example M was 2 and N was 2 so this number was 4 and this number was 4. But supposing – so they're the same. But supposing that this number was 8 and this number was 8 then this would be 16 and that would be 64. So usually you – most systems are not two state systems, usually. So there's much – what this is telling us is that in a general state of the system there's very much more information than there is in here. And why is that?

Because to specify a general state of the system you have to specify all the correlations between the subsystems. And there are a lot of possible correlations.

This is not a problem only for quantum mechanics. This would be a problem if we were doing statistical physics, classical statistical physics. Correlations are nothing to do, I mean not directly to do with quantum mechanics they're a logical problem that arises in all statistical inference. Also in the classical world. And correlations are very hard to handle in classical probability theory they're actually easier in this apparatus here because quantum mechanics pulls this amazing trick.

Correlated states of the system are obtained – are understood as quantum interference. Right a sum like this is a quantum interference between uncorrelated states of the system. When you're doing classical probability theory you aren't able to pull that trick and it's much harder to specify correlations. So correlations are important in both the classical world and the quantum world. But they're actually easier to handle in the quantum world than the classical world because of the strange way in which quantum mechanics compounds these amplitudes, does this quantum interference. So quantum interference is how quantum mechanics handles correlations and because it has its own completely unique way of handling correlations the results can be surprising, right. They can be ones that raise eyebrows and the Einstein Podolsky Rosen experiment is an example.

Let's try and pin these ideas a bit by looking at a concrete example of the H atom. So in the position representation what do we want to know? A complete set of amplitudes are going to be things like X. So this is – so let's make this the electron wave function. And we're going to have also – so we'll call this XE therefore and we will have XP times a big U. This will be a proton wave function. Right, which gives you the amplitude to find the proton at the point XP, this gives you the amplitude to find the electron at the point XE. And supposing these things have subscripts on them UI and UJ. So this might be the amplitude to find the electron at the point XE given – so this might be a UI given that the energy of the electron is EI. And this might be the amplitude to find the proton somewhere given that the proton's energy is EJ, say, right. Then what is the state of the H atom would be – sorry – XP.

So what is this? This is A's state of the hydrogen atom in which the proton has this energy, the electron has this energy and that gives me a state of the logically coupled pair of proton and electron. This as I say is not going to be very realistic state of the hydrogen atom because it's going to give us – this is going to give – this says that the electron and the proton are uncorrelated and I've just tried to persuade you that the electron and the proton are very strongly correlated consequently their wave functions can't – this isn't going to be a realistic useful wave function for hydrogen atoms as found in the Lab.

So what do we have to do? A more realistic state might be XE XP shall we say Ki for a new label which would be sum sum CIJ of XE UI XP big U J. But what are these? This is a boring function of X with a label I. There's a set of functions of XE which have labels I in return complex values and then this complex number is multiplied on this complex number which is a function of XP. A member of a family of functions with labels J. Here is an amplitude, another complex number. Add all these complex number together and you get this complex number and this. So any state of a hydrogen atom must be writable like this but realistic states are not writable like that because of this correlation of the proton and the electron.

Okay now we need to revisit collapse, whoops – of wave function – function. So what happens when we make measurements on compound systems? The – we know that when we make measurements – what happens when we make measurements on a single system and we have to extend these ideas. So suppose let's go back to our state of our system so we go back to the two states system, two two states system A and B. And consider this particular state – up si which is equal to A times plus up plus minus brackets B up plus C down. Supposing this is what we have. This is pretty much written down at random. It is a well defined state of the system because it's a sum of three of the four bases states that we were discussing, right. It's the sum of plus up, minus up and minus down. This is the amplitude that if you would measure A and you would measure B you'd find that A was plus and B was up. This is the amplitude for finding that A is minus and B is up etc.

But – I've written this. But this one down – this state is at it turns out entangled. That is to say you won't be able to write this as a product of a state A and a state B. So this is more realistic than the states that I was discussing before, okay. Okay now suppose we measure – so let's measure state – this is subsystem A. If we get plus then after measurement the theory says – right the dogma is – I'm not going to justify this I'm stating this as a conjecture that the state of the system is now goes to up si primed which is equal to plus up. So how does the system – let's just remind ourselves what collapse of wave function was all about in the one state system – in the one single system sorry.

If we had a single system we wrote up si was equal to the sum AN – let us say EN for example and we measured E and got the answer EM then up si went to the state M. Alright, after the measurement it was in this state. So by making – I'm stating that in this more complicated scenario where we have a two – we have a composite system we measure only one of the subsystems we get a certain answer it goes to that state which is consistent of what we had over there because we found the answer plus – so we threw away everything times minus but the... Whereas over there it's simply M the co efficient up there of plus was not just a complex number A which was given me the probability it was also times this state of B. And this state of B just gets copied down.

So what does this say? This – so this is what the theory claims is that that goes to that. It doesn't explain how this happens, this is the problem of measurement. But there's a physical implication of this which is that you're now a measurement of B is guaranteed to produce or to find up, alright. Because this thing is something times up there is now zero amplitude to find down. You're certain not to find down you're certain to find up. Even – right. If, on the other hand, we get minus for A then the new state is equal to minus – sorry, sorry the new state is equal to, yes – minus brackets B up plus C down property normalised so over the square route of B squared plus C squared. So this is what the theory claims that if you get the minus thing then your new state is essentially the co efficient of minus and minus itself all properly normalised.

And now, so if we get minus there is now uncertainty as to what the result of a measurement on B will be. So it's – so now measurement of B yields for example up with probability B squared over the square route of B squared plus C squared. So we now apply the same old rules about the probability of measuring – about the interpretation of the amplitudes right. Because we are certain to get minus if we measure A again but if we measure B we can get two outcomes either up or down and the probabilities are like that. So that's a conjecture, that's a statement, a theoretical statement about how the interpretation of the theory works and we just have to accept it and see whether it leads to proper experimental predictions.

So in our last minutes we have unfortunately a big topic to discuss which is operators for composite systems. So we've talked exclusively so far about the kets but we know that operators play a very important role with every measurable quantity there's going to be an operator and we need to know how this behaves. So we found that the kets of the subsystems were multiplied. This rule was inherited from the multiplication of probabilities of successive events. The operators add. So for example if we have two three particles if A and B are both three particles then HA is equal to PA squared the momentum of A squared over twice the mass of A. and HB, the Hamiltonian operator is equal to PB squared over 2 MB. So what's the Hamiltonian of the combined system HAB is equal to HA plus HB. In other words it's PA squared over 2 MA plus PB squared over 2 MB. And that's sort of saying the energies of the combined system is the sum of the energies of the individual bits.

How does the operator P – we now need to explain how an operator PA operates on one of these states here, okay. So when PA hits AIBJ what we have – so this is a state of the combined system and this is an operator which has to operate on the state of the combined system. And what does it do? It produces PA operating on AI which is a well defined state of A symbolically times BJ. If PB works on this thing it – PB ignores this. It passes through this as if PB was just an ordinary complex number and homes in on this, its target. So this is simply AI times PB BJ. This is a well defined state of B – gets to be symbolically multiplied by this well defined state of A and there you are.

So for example what would the expectation value AB IJ of H ABB in this case here? Let's just make sure that we get some sense out of this. Sorry AB IJ. So what does that mean? That means AI BI – sorry J, BJ brackets HA plus HB close brackets AI BJ. So this operator ignores that because it's a B operator and homes in on that. This operator operates on this and then we have the other things come in on the other side and this gives me AI P, sorry HA AI BJ BJ plus – so that comes from this, this and this. Because that passes through this A operator as if it was just – this was just a number and bangs into that. Plus correspondingly we're going to have AI AI BJ HB BJ.

This of course is going to be the number 1. This is going to be EA – the expectation value of the energy of A. This is the number 1 and this is the expectation value of B. So we find that the expectation value of the energy of the combined system is lo and behold the sum of the energy to the individual bits. I think that makes physical sense.

If it makes – that makes physical sense when the Hamiltonian takes that simple form. If it's just the sum of the individual bits but for example for hydrogen the Hamiltonian H is equal to P electron squared over 2 mass of electron plus P proton squared over 2 the mass of a proton minus the charge on the electron squared over 4 Pi epsilon nought X electron minus X proton in modulus. Alright because the energy of the hydrogen atom is the sum of the kinetic energy of the electron and the kinetic energy of the proton and an interaction energy of the two. Right because they electro statically attract each other.

So this is equal to a H electron plus H proton – these being the Hamiltonians of the three electron and the three proton plus an interaction Hamiltonian. And the thing about this interaction Hamiltonian is that it depends on operators belonging both to the first subsystem and the second subsystem. And the consequence of that is that HE comma H interaction commutator is not equal to nought. Because the PE, the electron momentum operator sitting inside here has a bone to pick with the electron position sitting inside here. And similarly of course HP comma H interaction is not equal to zero.

So without that interaction we would have that the – so what's the important point about this is that the Hamiltonian of the hydrogen atom does not commute with the Hamiltonians of the electron and the proton. You cannot know the energy so generically you do not expect to be able to know the energy of the hydrogen atom if you know the energy of the electron because they don't commute. And it's the interaction that stops them computing.

Well we're going to have to stop unfortunately at that point but we're pretty nearly done. I'll just write down one final statement which is that the operators of different subsystems always commute. Right, so for example, P proton comma X electron is precisely nothing etc. We do not have to worry about non vanishing commutators of operators that belong to different subsystems.

Okay.