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**Contributor** Angular momentum is enormously important in physics. For example it's central to all kinds of scattering experiments and scattering experiments are at - lie at the core of high energy physics. They play a very big role in condensed matter physics. Angular momentum plays a central role in the theory of the application of quantum mechanics to atoms to get atomic structure. So from the very beginning of the subject it played a very big role. And people write whole books, horrifically, they write whole books on angular momentum in quantum mechanics.

So we are going to have to spend a few lectures on it even though we won't – there won't be quite as much physical content. We're building foundations for later work is what I suppose I should be saying. But we will on Wednesday in the next lecture we will at least be able to do something interesting and useful with angular momentum so the outlook is not entirely bleak but I'm afraid today's lecture is a bit on the formal side.

So you will recall I hope at the end of last term we talked about – we talked about operators that generated translations. They turned out to be the momentum operators and we concluded that there must be operators that effect rotations. So there must be a unitary operator U of alpha which generates the state like the state you've already got. That generates the state – that's the same as the system you've already got except rotated by an angle mod alpha around the unit vector in the direction of alpha, right. There must be some unitary operator like this.

This is a unitary operator depending on a continuous parameter right you can either, you can shrink the angle of your rotation down to nothing continuously it's in that class of continuous of unitary operators. So it's generated – we can write it's the exponential of something or other by putting an I in there this thing becomes the hermitian operator. These J's – so and because there are three components of this vector alpha which describes the rotation that you're planning there must be three of these operators that generate these rotations and we're calling them of course GX, JY and JZ. I claimed – I said that they are the angular momentum operators but we haven't really done a great deal. We didn't do a great deal of that time to justify this claim.

Okay so we have those three operators they're the generators of rotations respectively round the X axis, the Y axis and the Z axis. Out of them because they're hermitian operators we can construct another operator called J squared as the sum of the squares of the operators and we have a set of four operators and we showed by considering what happens when you make rotations around different axis we demonstrated that these operators must have the commutation relations that J squared commutes with every – with all of them with all of JX, JY and JZ. And that these operators do not strangely commute with each other they have the commutation relations that JX, JY is IJZ and similar things which can be encapsulated in this way where `[[epsilon 0:03:18]]` and IJK is the object that keeps changing its sign. And is zero if any 2 of its subscripts are identical.

So what we want to do now. So that sort of showed the existence of these things. What we have to do in the next section is find out more about these operators and the eigen states of these operators. We need to justify the claim that these operators really are the angular momentum operators. We need to find crucially well it will turn out that the orientation of something like an electron, well indeed the orientation of any quantum system is encoded in the amplitudes to find the possible results, the possible eigen values when you make a measurement of JX, JY or JZ you will – there will be possible answers. There will be – you'll get a number which is – belongs to the spectrum of that and the amplitude for that event strangely encodes the orientation of the object, right. And we need to understand about that.

So what we want to know now really is what is the spectrum of these operators. You want to know what are the possible results of measuring J squared or JX squared or JZ squared or whatever, right. So this is what the next spectrum is about it's about the spectrum of J squared et al these operators. So since the JX, JY and JZ don't commute with each other there isn't a complete set of mutual Liedenkets of JX, JY and JZ. But there is a complete mutual set of Idenkets because of that commutation relation because J squared commutes with all of its subordinates. There is a complete set of mutual idenkets of J squared and any one of those and its conventional to study – to pick just at random that we choose to have mutual idenkets, J squared and JZ.

So it's just a convention that we choose JZ out of the three things connected to the fact that Z is the singular axis is the special axis in systems of spherical polar coordinates, alright. So in spherical polar coordinates X and Y have pretty much the same role in life but Z axis is special and that's why we choose this one, okay. So that's what we're going to do. So we're going to say "Look there must be some idenkets." We're going to label them by B to M. This is label is going to tell us how the thing responds to this operator concrete it's going to be this, alright. So obviously we're labelling the ket by it's eigen value with respect to J squared and JZ – oops JZ on B to M is going to be M B to M.

So the second label in this thing tells you how it responds to JZ. This is by definition a member of the complete set of mutual idenkets of this operator and this operator which the mathematicians have promised us exists, okay. Now we introduce some ladder operators we're going to follow a line of reasoning that's very similar to how we got the eigen values of the Hamiltonian of a simple harmonic oscillator. We're going to introduce J plus as JX plus I JY.

So this is a little bit analogous to when we introduced in the simple harmonic oscillator the destruction operator we said that A was equal to X plus IP. Similar again so because of this I this is not hermitian, it's not an observable, it's a tool of the trade. And correspondingly needless to say we have J minus which is equal to JX minus I JY and we also have that J plus dagger is equal to J minus, alright. So this thing here is the hermitian adjoint of that thing there because if you take the dagger of this equation this dagger goes into this because it's an observable. That goes to minus I and this goes into this. So these are tools of the trade.

Now we find what – now we ask ourselves what are the commutation relations. We have that J squared on J plus is nothing but J squared, JX plus I J squared, JY is nothing because this vanishes and this vanishes, right. So J squared commutes with J plus and of course it commutes with J minus as well, right, so this is plus or minus it vanishes. What does that tell us? That tells us that if you've got – if you take J plus of beta M you use this non travil operator on this state, you get some other state. What can we say about this other state well one thing we can say is that J squared applied to it because you can swap these two over is the same as J plus beta, beta M. So you swap these two over then J squared looks at this and says aha that's my identiket out pops a beta. This is a mere number can be popped over here is equal to beta J plus beta M.

So when you use J plus on this eigen state of J squared you get another eigen state of J squared for the same eigen value, right. Encouraged by that the next thing to do is to have a look at JZ on J plus beta M. Now when we swap these two over well we want to swap the two over but of course we can't so we do the usual business J plus, JZ I've swapped them over and then add in what we should have and take away what we're not entitled to JZ, J plus commutator brackets, brackets beta M.

Now this we found what this thing was. We found that…oops sorry we didn't, we didn't sorry… I'm getting ahead of myself apologies. Right so I mean that's all we need to. Okay well we're going to find out what this is, we're going to find out what this is that's the next thing we have to do. Alright, so what is JZ, J plus. Well it's JZ, JX plus I JZ, JY. This is minus I JY from the rule given way up there and this is minus I so this is going to be I, this I minus coming up another I J X. Again from the rule above. So this is going to be minus, sorry this is going to be J plus because this is – these two I's are going to make a minus sign, cancel this, we're going to have JX – it has to be plus.

So what the hell's gone wrong here is I've goofed presumably in that XY – yes I've goofed in that, sorry. I'm always bad at this cyclic ordering. So it is equal to J plus. So we take this important result, we stuff it in there and we have that JZ on J plus beta M is equal to – so this is going to be J – this is going to be J plus and JZ working on that is going to produce an M times that. So we're going to have an M plus 1 times this.

So what does that show, that shows that when you apply J plus to this object you get a new eigenket of this operator one which has this for a eigen value. So what – let's write that down it says that J plus on beta M is equal to M plus 1 – sorry is equal to some amount of which we will call alpha plus the state beta and M plus one okay. So the point is that what goes in here is the eigen value of this thing with respect to JZ. So this thing here, this thing here turns out to be – this shows that it is an eigen ket of this operator with this eigen value M plus 1. So that's what should go in there.

And this is some normalising constant. So what have we achieved when we applied J plus to this state JM we made a new state with the same total amount of angular momentum, the same response to J squared but the amount of this parallel to the Z axis has increased. So we have reoriented our system, right we have here a spinning top well okay so angular momentum along here and we've moved it a bit towards the Z axis. That's what J plus does. It realigns the angular momentum that you've got. Strictly speaking it makes you a new state and this new state has the same angular momentum as the old state but more of its parallel to the Z axis.

Okay we could repeat all this stuff. I recommend that after the lecture you do repeat all this stuff using J minus and you will find that J minus on beta M is going to equal some amount not to be determined, not known yet of beta M minus 1. It does the reverse trick, it moves it away from the Z axis or if you like towards the minus Z axis. So showing this is precise repeat of what was done up there except every plus sign gets turned into a minus sign.

Okay. Now we have that – the expectation value of for example JX squared is equal to JX up si for any state up si, so take any state up si and work out this expectation value of JX squared. It's JX up si mod squared right because if you take the mod – if you take the mod square of this what you're taking the hermitian adjoint of this which is that, the hermitian adjoint of JX which is JX itself and multiply it into this so you end up with this. And this is clearly, this is the length squared of a vector so it's greater than or equal to nothing for all up si.

So let's ask ourselves about JM J squared JM. That's clearly equal to beta because J square on to, sorry, M beta M, beta M. So J squared on this produces beta times this, this is correctly normalised so we get beta. But this can also be looked as beta M JX squared beta M. So it's equal to this plus beta M JY squared plus beta M JZ squared. But this, this last one here is clearly M squared because J – one of these J's looks at this and produces an M times beta M and the other one then looks at that and produces another M times beta M and we end up with just M squared.

So what have we got, we've got that beta is equal to – well what shall we call this we'll call this A and we'll call this B is equal to A plus B plus M squared where these numbers are greater than or equal to nought. In other words we concluded that beta is greater than equal to M squared so there's a problem. We've got an operator J plus which can make us a new state with M increased by 1 which has – but has – but this new state has the same value of beta. So apparently we can make states with bigger and bigger M for the same beta and that we've just shown mathematically that that's absurd, physically it's absurd because I'm saying that I've got a fixed amount of angular momentum and J plus just moves it towards the Z axis. Well eventually you'll have it parallel to the Z axis and it won't be able to increase M anymore.

So what truncates this something has to give and what – just like the harmonic oscillator what gives is that eventually – so series of states of bigger M truncated at beta M max for maximum value of M such that how does this happen? It happens because when we use J plus on this state we get exactly nothing. So what does this mean? This implies that alpha plus equals nought in this particular case. That's the only way we can be stopped from making states of bigger and bigger M and it's clear we have to be stopped, so we are stopped in this way. So what we have to do now is look at the mod square of this, of this state and show that it's zero. So we have nought is equal to mod so the mod square of this is going to be this hermitian adjointed times J plus. Sorry J plus dagger which is J minus times J plus times beta M max.

Right so this thing here, this is J plus dagger which is appearing here and I pointed out earlier on that J plus dagger is J minus. So let's have a look and see what we've got here by staring inside. So this is going to be – I don't need the mod square that's already taken care of. So this is beta M max, JX minus I JY, JX plus I JY close brackets beta M max. So we multiply this stuff out and we get JX squared plus JY squared and then we get, we have a minus I JYX and a plus I JXY. So we have plus I commutator JX, JY.

Well when we've got this much of J squared you might as well have the whole of J squared so we write this as beta M max J squared minus JZ squared alright. So we add a JZ squared and take it away again. And this of course is I JZ so along with that I we get minus JZ beta M max. And now we're in clover because we know what every single one of these operators produces when it bangs into that. So we can evaluate this.

This of course produces a beta so this is going to be – this is going to produce a beta times this thing. Then this thing will meet this thing and produce 1 so I only need to write down beta. This JZ will produce an M max times this thing which will then bang into this thing and produce a 1. So I have a minus M max and this one is going to be produced M max squared also with a minus sign. So in fact let me write this as – oh never mind M max squared. So this is more conveniently well alright so what do we have – we have that nothing is equal to this stuff from which it follows that beta we've discovered now what beta is in terms of M max it's equal to M max brackets M max plus 1.

So if we apply J minus to beta M I claimed that this was alpha minus beta M minus 1. So M will start. Let's imagine M starts off positive as we take units from it it's going to get smaller and if we keep going presumably it'll become negative and M will start to be a growing, a negative number of growing magnitude. But we still have this condition that M squared is got to be less than M squared has got to be less than beta. So this series of operations has got to terminate as well. So series of kets with ever smaller M has to stop. So there must be a minimum value of M which we imagine will be negative so we're going to have that beta M min times J well I should write differently. I should say that nothing has to equal the mod square of J minus applied to beta M min. And when we expand that out we'll – they'll be other things happen and let me – so in other words nothing is going to be beta M min J plus J minus beta M min. That's awfully similar to what we had here when we had a J minus, J plus. So you can see that it's going to produce the same stuff except that the sin of the commutators are going to be different otherwise everything will be the same. So this is going to be nothing is going to be beta min J squared minus J Z squared plus J Z which is going to lead to the conclusion that beta nothing is going to be beta minus M min squared plus M min. In other words, beta is also equal to M min M min minus 1.

So we have a relationship here between beta and the largest value that M can take and between beta and the smallest value that M can take. Now we could, well we can – we can from these – between these two equations we can eliminate beta and learn that M min squared minus M min which is this is equal to beta or minus beta equals nought but minus beta is the same as minus M max, M max plus 1. So we have this equation and this could be thought of as a quadratic equation for M min in terms of M max, right. So this is a quadratic equation and it tells me that M min is equal to minus B, well B is minus 1 so it's equal 1 plus a minus the square root of B squared minus 4AC. A is 1, C is minus this stuff. So plus 4 M max brackets M max plus 1. All of the two.

Looks ugly but actually it's very beautiful because this is going to be a half of 1 plus or minus the square root of. This – well let me write down what it is and you can tell me whether you agree with it. It's M max plus 1 squared. If you square this stuff up you get 4 M max squared you also get 4 M max from the cross terms, 2 times 2 makes 4. So that's that and that. And you also get a 1, that's that. So we can extract the square root, right because we've got the square root of a square. So we have plus or minus this.

M min is obviously smaller than M max so the plus route can be ignored because that would tell me that M min was bigger than M max. So only the minus route is wanted and you soon find that that is equal to minus M max. So there's a biggest value that M can take and there's a smallest value that M can take and we've shown that that's minus the biggest value. In other words we've got a picture like this. We have a biggest value here then we have a next value, then we have a next value, then we have a next value and suppose that this is the end then zero lies, so this is a plot with M going up here. So here would be zero say and in this case this would be a half, this will be three halves, this will be minus a half and this will be minus three halves.

Or it might work out like this that we'd start – but the key thing is we can start slightly higher up and then we would have this one, this one and this one and this one. So if we started at 2 we could have 1 nothing minus 1, minus 2. These are the possibilities. But the key thing is that I know that in an integer number of steps here 3 steps I could go from the biggest value to the smallest value. Here there are 4 steps 1, 2, 3, 4. So the key thing is that twice M max is an integer.

Now we could carry on talking about beta and M max but it's extremely boring and nobody does that. What they do is they use a new notation. They say that – well the new notation is that J is what you mean by M max. The biggest value of M is called J, little J. And what have we got, we've got that beta is equal to M max, M max plus 1. That's on the board just here is therefore equal to J, J plus 1. And we know that 2 J is an integer in other words J is a half integer or it may be an even number of half integers in which case it's an integer itself or it may be an odd number of half integers.

So in this left hand column J is a half integer. All the values are J – sorry J is a half integer. Consequently all the values of M are half integers. In the right column J happens to be an integer and therefore all the values of M are integers.

Therefore this beta number is sometimes an integer so if J is an integer this is an integer. For example if J – a possibility is that J comes out being nought in which case beta's nought. Or J might come out being 1 in which case beta would be 2 or J might come out being 2 in which case beta would come up being 6. We have a sort of funny selection of integers. But worse than that when beta is – sorry when J is a half integer the values of beta are really quite weird so we don't use beta as the label. So we re-label beta M to JM. Instead of using as the label in here that tells you how this state responds to J squared. Instead of using the actual eigen value you use this number which is either an integer or a half integer from which you can work out this eigen value because this eigen value is J, J plus 1. That is to say we have that J squared on JM is equal to J J plus 1 JM. And we have that JZ on JM is equal to M of JM. This is the new notation universally used.

So we change notation only because we've discovered that numbers beta are themselves rather unpleasant and don't make for handy labels but they are related through this equation to something that's very simple which will be an integer or a half integer and moreover tells us immediately what the largest value of M is that you were allowed to have.

So we have, if J equals 2 there are 5 states there is 2,2 2,1 2,0 2, minus 1 and 2, minus 2. Now what does that mean, what statement is being made physically it's being said that if my pen has 2 units of angular momentum well it has J equals 2 which means as I've said it has strictly speaking J squared has an eigen value of 6, right but if – we call that 2 units of angular momentum. It has 5 possible orientations, right. This one, this one, this one, this one and this one only 5. This is what they called space quantisation when Stern and Gerlach discovered this experimentally. I think it's a terrible term right it's not. I wouldn't call it I think it's – no I think it's a very bad idea to call it space quantisation but I just tell you historically that's what they called it. But this is the bizarre conclusion that we have a discrete set of orientations anyway being possible for a pen with that amount of angular momentum.

If J is a half then what do we have we have a half and a half and a half and minus a half and that's it, only two states. So that's why we've been talking about electrons and things, objects with angular momentum with spin a half, half a unit of spin angular momentum like electrons, protons, positrons etc as the archetypal two state system because there are only two possible orientations. Now this is very misleading right but I've already given health warnings on this but the naïve interpretation is that your spin a half particle, your spin a half giro has two orientations, this one and this one and nothing in between allowed, okay.

So that is a grossly oversimplified picture which leads to misunderstandings but it gives us a bit of orientation and people often do think in those terms. In the three halves case we would have three halves, three halves, three halves, one half, three halves minus one half and three halves minus three halves. We would have four possible orientations. We'd have this, this, this and this never pointing horizontally etc etc etc, okay.

Almost done. Let's have a look at the effect of rotation around the Z axis. Okay so up si goes to up si primed which is U of Alpha of up si. So any, the... These angular momentum operators came in as the things you put in an explanational in order to generate a rotation, the unitary matrix that makes you a new system which is the old system rotated. So we want to see what we get now. So let's see what happens when we rotate a state of well defined, one of these eigen states here right. So let's do E to the minus – so if we go about the Z axis then alpha only has a component in the Z direction and this becomes and it has a magnitude Fi so this becomes E to the minus I, Fi is the rotation angle JZ and let's use that on one of these JM states.

Well this is a function of an operator used on, it's a function of a operator so by the definition of a function of the operator it has the same eigen states as the operator whose function it is. So this thing is an eigen state of this operator and the eigen value is the function on the eigen value. So this is E to the minus I Fi M JM. So one of these states of – one of these eigen states here when you make – when you rotate it using this rotation operator produces you the same state multiplied by this phase factor.

Okay so if we rotate through 2 pi. If we rotate the thing completely around. So if we put Fi to 2 pi we are looking at – what are we going to call this? We're going to call this up si primed, say, right. Up si primed is going to be E to the minus 2 pi I M. Well may be we should say 2M pi I times what we first thought of.

If M is an integer then – so this E – then this is going to be a number 1. So this is equal to JM if M is an integer. But it's equal. To minus JM if M is a half integer as we know it can be. So we have the surprising result that if you rotate, a system with half integer angular momentum completely around, complete through an entire rotation its state doesn't return to its original state, it returns to minus its original state. And this seems strange to us because we don't have any concrete experience, we have no experience of this kind of thing for the following reason that particles which have even though – yes okay particles which have half integer J. Well particles are described by fields. Particles that have half integer J are described by fields whose value never becomes, this is a result of quantum field theory, whose value never becomes large compared to the quantum fluctuations in the field, the quantum uncertainty in the field.

So the values of these fields never become significant and we have – these fields never enter classical physics. So the direct field whose excitations are electrons and positrons are – is not something that's part of classical physics. It's a part of the vacuum just the same as electro magnetic field or the gravitational field but it's never excited at a macroscopic level so it doesn't enter classical physics.

So we have no experience as classical beings within classical physics of the fields associated with these half integer values of M and therefore are unaware of this fact that if you turn the thing completely around it changes its sign. And the fields we do have experience of the electro magnetic field and the gravitational field belong to integer values of M, the electro magnetic field has M, well has J rather equal to 1 and the gravitational field has J equal to 2 and therefore these fields don't manifest this strange behaviour.

Well I think that is the right place to stop even though it's a little early and we will look at the rotating molecules as a physical application on Wednesday.

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