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Contributor So the big idea I introduced yesterday was that of a quantum amplitude – a complex number whose mod square gives you the probability for the outcome of some experiment. Some measurement.

And we introduced the concept of a complete set of amplitudes – quantum amplitudes – so that if you knew these many – all the amplitudes in a complete set – then you could calculate the amplitudes for any experiment that you might conceive. Any measurement that you might make.

And I made the point that quantum mechanics is all about going from the amplitudes in some complete set. Calculating these other amplitudes for the outcomes of other experiments.

So, this is – there is a very powerful analogy here between – so our knowledge of the state – the dynamical state – of our system is encapsulated in the values taken by these complete sets of amplitudes. So it's some series. It's some set of complex numbers.

And there's a very good analogy here between the way that we identify points in space and the co-ordinates of vectors. So we can use many different co-ordinate systems and many sets of numbers to identify one and the same point in space. So points in space are a primitive notion. And the sets of three numbers we used to identify them depend on preference. And you might use – there are many co-ordinate systems. You might use many different Cartesian co-ordinate systems. We might use polar co-ordinates we have. And the co-ordinates we use to identify a given point depend on the problem we're trying to solve.

It may be most efficient to use spherical co-ordinates. It maybe most efficient to use a particular Cartesian co-ordinate system or whatever.

So we want – and we find it very useful – to have the concept of a position vector R which we understand to be x comma y comma z. It's set of three numbers. But it's more than a set of three numbers. It's really an equivalence class of sets of three numbers. Because every different co-ordinate system would have a different set of three numbers all for the same point.

So Dirac introduced the concept of a ket. So this symbol effectively characterises the – this symbol stands for the state of our system. The dynamical state of our system. And you can think of it – it symbolically stands for A1 comma,

[[Break in Audio]]

(Laughter)

comma, A3 comma, A4 comma, dot, dot, dot right. So we don't know how many quantum amplitudes we need in order to characterise our system. So it just goes, dot, dot, dot.

[[Break in Audio]]

But the power of the notation is the power that we get from position vectors. Instead of writing all this – if we write all this stuff down, then we are committing ourselves to a particular co-ordinates – to a particular co-ordinates system if you like. To a particular set of complete amplitudes.

Whereas what we really want to do is focus on the dynamical state of our system. This is the dynamical state of our system. We might find it convenient to use the amplitudes to find the different possible energies. We might find it convenient to use instead the amplitudes with the different possible measurements of the momentum or the position or whatever. We leave that flexible by using – excuse me – there we go – by using this symbol.

[[Said 0:03:54]] ket – and of course, sorry, that is the back end of bra-ket. We will have bras in a moment.

Okay. Now we know what it is. We can – if we have got two kets. Supposing this stands for – this is another dynamical state of the system and let it be defined – let it be ima- in some particular system, let it be these numbers, B1, B2, B3, etc. Then, because we know what it is to add amplitudes. Indeed we know we're under orders to add amplitudes when something can happen by two different routes, it makes sense to define the object. We know what this object is. It is A1 plus B1 comma, A2 plus B2, 2 comma and so on.

So if you add two kets, that says the dynamical state of the system, which is described by the amplitude, the first amplitude being the sum of the amplitudes from the individual bits. The second amplitude being the sum of the amplitudes – the second amplitudes – for the individual bits and so on. Right?

So just as you add two vectors – if you add two vectors, you add the x components and you add the y components and you add the z components to make a new set of three numbers. That's what we do with ket.

So we know how to add kets now. And we also know what it is to multiply kets. We can define a new ket psi primed being – which we write like this – alpha upsi, which is just some complex number. We define this to be the ket alpha A1 comma, alpha A2 comma and so on. In other words, if you multiply a ket by some complex number, alpha, what you mean is the dynamical state of the system that you would have which has amplitudes alpha times the original alpha amplitude in every slot.

So we know how to add these things. We know how to multiply these things by complex numbers. It follows that kets form a vector space. So I guess you've been – you've encountered this idea with, in Professor [[Eessler 0:06:18]]'s lectures, right? That the elements of a vector space are, for a mathematician, they are nothing but objects which you can add and objects you can multiply by numbers. Either real numbers or complex numbers at your discretion.

So that kets form part of a vector space – we'll call this vector space big `[[B 0:06:36]]`.

You from those lectures I hope know that what you get f-

No, let's, yeah, from those lectures I hope you have met the idea of a basis. A set of basis kets. What is a set of basis kets? It's a set of objects 'i' like this which is such that any ket can be written as a linear combination to whatever you need. It's a set of kets such that any ket – for example, the dynamical state of our system – can be written as a linear combination of these kets. Right?

Then we have the idea of an adjoint space. I hope I am just reminding you of stuff that you've already met. So if we consider the linear – we are going to be very interested in the linear complex-valued – complex-valued functions on kets.

A mathematician would say on V – functions on the elements of V.

So, and you might imagine – traditionally you would say, okay, F of upsi is a complex number. The complex number in question is going to be the amplitude. The reason why we care about these functions is because they're going – these complex numbers – are going to be the all important amplitudes for something to happen. For something to be measured. Right?

And that's – you know, we're completely focused. The whole – all this mathematical apparatus is only there to help us to calculate these amplitudes. Because, if we can calculate amplitudes, we can take the mod square and we then have a prediction for what some experiment is going to – a probabilistic prediction for what some experiment is going to yield.

Okay, so we're interested in these complex-valued functions. I'm just saying that they're going to turn out to be the amplitudes. I'm not establishing that at this point. And the thing is, we don't actually use this notation. The notation we use is this. But these mean the same thing. Bracket opening – sort of angular bracket opening this way F upsi. This thing here means the function F evaluated on upsi means that it is a complex number. It is going to be interpreted as an amplitude for something to happen.

And this gives us the idea of saying that F, which – so this thing is a function. A linear – complex-valued function – is called the bra. The bra F.

So we've got kets which define dynamical states of our system. And we've got bras which are functions on the dynamical states of the system which extract the all important amplitudes. The kets for a vector space. Because it's a vector space, it must have basis, like that up there. And the bras also form a vector space, as I hope you've discovered in Professor Essler's lectures. So the bras form the adjoint space. Often called `[[V-primed 0:10:22]]`.

Why do they form a vector space? Because I know what it is to add two bras. If I – given – if you give me a bra F and a bra G, I can form a new bra – let's call it H for originality. Right? What – in order to give meaning to this, I need to know what H does – what H does to any state upsi. I want to know – a function is to find where the value it takes on any ob- on any possible argument.

So I need to know what H upsi is. What number that is. And I define it to be F upsi plus

G upsi. Which, of course, is a perfectly well-defined expression because this is complex number, this a complex number and we all know how to add complex numbers. So this is the definition of the func- of the bra H.

So I know what it is to add two functions. And, of course, I know what it is also to multiply a function by some constant thing. So I define the ket G-primed – meaning alpha G – by the rule G-primed of upsi is alpha G of upsi.

Okay, so again this is perfectly well-defined because that's just a complex number and so this multiplication is well-defined. So now I know what G-primed – what value it takes in every upsi.

So this is the point that this is the basic principle that establishes that the functions – the linear, complex-valued functions on a vector space – form a vector: the adjoint space. And we're going to be working extensively with both the kets and the bras.

The only other thing that we need to remind ourselves is that the dimension of the adjoint space is equal to the dimension of the space itself. And so if we – and how do we define this? We have a corres- well we prove -

So if we're given a basis kets i, for each one of these we define a bra and we do it as follows. We say that the bra J is the object – is the function on the kets – such that this complex number, Ji is equal to delta iJ. So, in other words, it's nothing if J – the label J – is not equal to the label i. And it's 1 if the label J is equal to the label i. Right?

So this equation defines J. The bra J. The func- So that we're saying that, for example, 2 – the function 2 belonging to the second ket in our basis is defined – this is a function – and it is defined as such that 2 on 2 is 1 and 2 on anything else equals 0.

So that is a perfectly good rule which defines the value that the function J takes on every element of the basis and again from Professor Essler's lectures, I hope you're aware and can show that if you know what a function takes in every element of the basis – a linear function takes in every element of the basis – you know what it takes in every ket whatsoever.

So there's one final thing that we want to do in this abstract area. We want to say, supposing upsi is equal to the sum Ai of – so we take a state of our system and we have it as a linear combination of the basis states. Then we define a function – this is a funny part, right? So so far I hope, I think, everything's been, I hope everything's been fairly straightforward. But now I'm saying associated with the state of our system, I wanted to find a function on states.

And the function in question is defined by this rule. That it's Ai complex conjugate times i – the bra i.

So, given that my state of my system is a certain linear combination of the basis states, I'm saying that the function associated with that state of the system is a certain linear combination of the functions – these functions – which are associated with the basis states.

Why do we do that? One reason we do that is in order that we can evaluate this important number, upsi, upsi. So let's have a look at that number. That is the sum – I write this out as a sum Ai*i `[[summed over i 0:15:39]]`. And then I have to write this one out as a sum AJ of J. So I'm summing over J. These are just dummy labels, right. So I'm entitled to call one J and one i. So it's a sum over J. Is 1 to however many we need. And i is 1 to however many we need.

This is a linear function. Right? We're evaluating this linear function on this dirty great sum. But because it's a linear function, the dirty great sum can be taken outside. So I can write this as the sum over I and now J, being 1 to whatever it is, of Ai*AJ of iJ. And there I've used the linearity of the function i. And now I use the fact that this is by definition of this function, delta iJ. So it is nothing unless i equals J. So now let's do the sum over J for example. As I do the sum over J, I will get nothing here except for that particular J which is equal to i and then this will become 1.

So this becomes the sum of Ai*Ai. In other words, it becomes the sum of Ai mod squared which now – that's just mathematics. Now we're back to physics. This is an amplitude to find – this is, this should be an amplitude Ai – a quantum amplitude. And we're taking a sum of the mod squareds of the amplitudes. So this is the sum of the product, sorry, of the probabilities. So that should be 1 because the probabilities should all add up to 1.

So my states – I would like my states to have this normalisation condition. This is proper normalisation. Is that any – the state times its bra should come to 1. Not any other complex number. That particular complex number, 1.

Okay, so that's the basic principles of direct notation. Now let's just talk about the energy – Let's have a look at this better understanding of what this physically means by having looking at energy representation.

So supposing we, in certain circumstances, for example if you've got a particle that moves in one dimension, then it's possible in some trapped in some well then it is possible to characterise the dynamical state of this system simply by giving the amplitude to measure the possible values of the energy.

So a complete set – so this is not always the case – but for a one-dimensional particle, a particle trapped. This is a very idealised situation but never mind. Trapped in a one-dimensional potential well. We will see that. And I'm asserting for the moment that the Ai form a complete set of amplitudes where Ai mod squared is the probability of measuring the ith energy. The ith allowed energy.

Right, so the energy in this case, when we have our particle trapped inside a potential well, has a discrete spectrum. Remember we introduced the idea of a spectrum. Those are the possible values of your measurement. You can only measure a discrete set of numbers. They're called ei. There's a probability that, if I would measure the energy, I would find the energy to be ei. But that's this mod square. And a complete characterisation of the system – complete dynamical information is provided by knowing not only these probabilities but actually the amplitudes themselves.

So you can think of upsi as a vector formed by these amplitudes.

Now, let's write that upsi – the state of our system – is equal – Let's be given some basis and let's write that it's equal to Ai i summed over i.

So out of these complex numbers which we know and some basis – any basis – we can write a symbol like this. That's just a repeat of what we've already done.

And now let's ask ourselves, what are the meaning – what's the physical meaning of these states?

These are – this is expressing my actual state of the system as a linear combination of some states of the system that we've conjured out of nowhere. Right? But each one of these is, according to our formalism, corresponds to a complete set of amplitudes. It's a state of the system. Now let's find out what these ones mean in this context.

Suppose we know the energy is actually e3. So that implies that A3 is 1 and Ai equals 0 for i not equal to 3. So supposing we happen to know that the energy is e3. Then the amplitudes must be like this. And what does that mean?

That means upsi – the state of our system – is actually equal to 3. Because in this sum, there's only going to be one non-vanishing term. And that will be A3 – namely 1 times 3.

So that tells us that this state 3 is actually the state of definitely being having energy e3. And similarly for all the other ones.

So a better notation, or a clearer notation, is to write, to rewrite that in a clearer notation as upsi is the sum i of ai times ei.

This makes it clear what we've just established that the thing i is actually the quantum state of definitely being having energy ei.

So we've discovered the physical meaning of those abstract basis vectors. When these are the amplitudes to measure the different energies.

And this is called the energy representation, right? This is the energy representation. This is when we express the state of our system as a linear combination of states of well-defined energy.

This representation is – it plays an enormously important role in quantum mechanics because it's how we – it's by going to this representation for mathematical reasons – going to this representation is how we solve the time-evolution equation. In other words, we solve the quantum analogues of Newton's laws of motion.

It's also, as we will find, a very abstract representation in the sense that, and this may surprise you, no physical system ever has well-defined energy. So these quantum states are in fact unrealisable in the real world.

So this expresses a realisable state of affairs as a linear combination of states that you can never actually find anything in. But it's of enormous technical and mathematical importance.

Let's talk now about something – we'll come back to the energy representation later on. But now let's move straight on to another illustration which is back to spin-1/2.

So I said that elementary particles are these tiny gyros that the rate at which they spin never changes. But the direction in which the spin is oriented does change.

I made the point yesterday that the – though you can know for certain the result of measuring the spin in one particular direction. For example, the component of the spin parallel to the z axis. You cannot know the direction in which the thing is spinning. Because even when you've measured the component parallel to the z axis with precision, you're in deep ignorance about the value of the spin parallel to the x axis or the y axis. You only know it does have spin in those directions. But you do not know the sine of this. You do not know how much spin is along x or along y.

But a complete – So, for S. So if we measure the spin along the z axis, and I'm going to say that this is now plus or minus a $\frac{1}{2}$. Now yesterday I had an h bar here in some sense. I was using a slightly different notation but I had an h bar there. I want – the angular momentum – h bar has dimensions of angular momentum.

So, the angular momentum, what this means is that the, if Sz is plus a $\frac{1}{2}$, that means the angular momentum in the z direction is plus a $\frac{1}{2}$ h bar.

But it turns out to be convenient to leave off the h bar when talking about the so-called spin of Sz. Partly because you'll see that spin in quantum mechanics is really has a slightly dimensionless being. And partly because in – we don't want to write any more h bars than we have to. It's just, you know, it's economical.

So physically, the angular momentum is a $\frac{1}{2}$ h bar. But it's more convenient to write that Sz, this abstract thing, the spin, is plus a $\frac{1}{2}$ or minus a $\frac{1}{2}$.

So what do we have? We have two states. We have a complete set of states formed by plus and minus. Okay? So this is the state in which I am certain, if I measure the spin parallel to the z axis, that I'm going to get the value a $\frac{1}{2}$. And this is the one where I'm certain to get minus a $\frac{1}{2}$.

And the statement that that's a complete set is to say that any state of my electron, or whatever, could be written as A plus plus – actually maybe it's better to write it this way. A minus minus plus A plus plus.

So since this is a nice easy case, there are only two components to our ket. A minus and A plus. And in just the same way that I might, in ordinary vectors write that R is equal to vector A, let's say. B perhaps is better. B is equal to Bx ex plus By, ey plus Bz ez. Don't need a bracket do I? No.

Where here I've got three real numbers, Bx, By and Bz, which are the components of B in some particular co-ordinate system. So here I'm saying the state of our electron could be written as a linear combination of this basis vector and this basis vector. So these kind of map across here. But this is a simpler case in so far as I've only got two components – A minus and A plus – rather than three components. So that's the analogy.

Okay.

Now we need to anticipate a formula – so what I claimed was earlier was that if you know what A minus and A pluses are – what those amplitudes are to find the spin in the z direction, either up or down – then you can calculate the amplitude to find the spin in another direction. Either parallel to that direction or anti-parallel to that direction. Okay? That's what I claimed.

And now I'm going to quote a result which we will –

`[[Audio falls silent at 0:28:55]]`