Title	*021 Even further Orbital Angular Momentum - Eigenfunctions, Parity and Kinetic Energy*
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Contributor Okay so yesterday we slaved over a hot chain rule in order to recover this formula here. So what we're trying to do is find the wave functions that represent the states of well defined, orbital angular momentum. And I explained what the strategy was for doing that and that strategy involved knowing what these differential, what these operators are as differential operators in the position representation and some rather tedious chain rule work was required in order to extract this formula here and I finished with a triumphal acquisition of this formula here. And I merely stated that this was what you got if you pursued the line of argument to find L minus.

Okay so we're now in a position to find these wave functions let's call it upsi LM this will be some function of R theta and phi because we're looking at this in spherical polar coordinates so this of course is R theta and phi LM. Now the radial dependence of this wave function is going to be completely unspecified because we're only going to require what we're going to require is that LZ on LM is equal to M LM. It's an eigen, this thing has got to be an eigen function of this operator with the eigen value M which we now know to be an integer. And we are going to require similarly that L squared on LM is equal to L L plus 1 of LM. And we will find – we haven't yet calculated this operator as this what L squared looks like is a differential operator we will get to that. But it will also turn out to involve only derivatives for sector theta and phi. So these operators none of them involve anything about radius and so this function is an arbitrary function of radius and all we're going to be able to discover is what it's angular dependence is by imposing these requirements here.

Okay so what can we say, we can say first of all this equation is going to imply put into the position representation it says that minus I DBD phi of upsi is equal to M of upsi which we of course immediately recognise as telling us that upsi at R theta and phi is equal to E to the IM phi times upsi at R theta and nothing if you see what I mean. And nothing and nothing and nothing.

So there's some kind of a constant here right. This thing doesn't depend on phi if you differentiate this with respect to phi you bring down an IM. The I's make another minus sign that cancels this and you end up with M times whatever it is. So what we know is – this should have it's subscripts I suppose. So what we know is that upsi LM is equal to some function of R and theta times E to the I M phi. I guess we kind of already knew that.

Now we're going to – oops. And we're going to well, yes. We're now going to impose the condition that L plus on upsi LL is equal to nought because this operator would create a state in which M this is – we've put here the value of M equal to L which is its largest value this would try and make a value, make M even bigger than L and we know that's not possible. So we have this equal zero. So copying down what that is, turning this equation into the position representation E to the I phi – sorry E to the… yes E to the I phi D by D theta plus I cotangent theta D by D phi operating on E to the I L phi times this function K which depends on R and theta especially it depends on theta, the R dependents we don't care about because we've got no differential operators here with respect to R.

So this term looks only at that and brings down an I L, right. So what's that – sorry this has got to equal zero so we've got two terms. This DBD theta terms looks only at that. So we discover that DK by D theta minus Cop theta, sorry L Cop theta we're bringing down an IL K equals nought. Then there's a factor of E to the something or other phi which we can cancel away it's not interesting.

So here we have – this is a… This is a linear first order differential equation – the friendliest kind of differential equation. So we saw that with an integrating factor the integrating factor is E to the integral of this here. E to the integral of minus L – whoops minus L cot theta D theta. But I think that the integral of cot theta D theta is log sin theta so this becomes E to the minus L log sin theta or E to the log of sin to the minus L theta or simply sin to the minus L theta. That is the integrating factor of this equation. In other words the equation states that D by D theta of the integrating factor which is sin to the minus L theta times the function is equal to nought. In other words this thing is equal to a constant. In other words K is equal to a constant which is obviously going to be some kind of normalising constant times sin to the L theta. And we have discovered this constant in principle depends on R, right. It's allowed to – you can have any R dependents you like so what we've discovered is that if si LL is any function of R you like times sin to the L theta E to the I L phi. This is an important kind of result.

And now we're in a position to calculate anything else because if we want to find what upsi L L minus 1 is then it's equal to L minus divided by inappropriate normalisation factor which happens to be L, L plus 1 minus L, L minus 1. Remember these ladder operators come with square root normalising factors that was the case in harmonic oscillator that's the case also with the angular momentum operators operating on upsi LL which we now know what it is. We now know that it's sin to the L theta E to the I L phi times the unspecified function of radius. And this L minus – let's roughly speaking put in what it is, no may be we do it on the next board because we want to be able to see those magic formulae, right there they are.

This tells me that upsi LL minus 1 is equal to. I think this is just a square root of 2L. So it's function of radius over the square root of 2L all being well times E to the minus I phi times there's probably an overall minus sign coming from that formula at the top there D by D theta minus I cot theta DBD phi working on the function we first thought of which is sin to the L theta E to the I L phi. And what are we going to get?

This DBD phi will again bring down you know an L etc and then this expediential will take 1 off that. So we'll end up with something that goes by E to the I L minus 1 phi. This will differentiate sin to the L and produce L sin to the L minus 1 times a co sin. This cotangent multiplying that because this is cos over sin will again produce me a cos times sin to the L minus 1. So the whole thing is going to be minus unspecified function of radius over the square root of 2L times there's going to be – everything's going to go like E to the minus I, L minus 1 phi. And then from here we're going to get an L from here differentiating that we're going to get an L well there's going to be a factor sorry of sin to the L minus 1 theta cos theta and how many of them. From here we'll have an L and from here we will have – we're going to bring down an L well a minus – sorry an IL. They will cancel – so I think it'll be plus another L of the same stuff. So you see that we have something like the square – minus the square root of L over 2 whatever your unspecified function of radius was E to the minus I or minus 1 phi times sin L minus 1 co sin theta.

And we could now apply L minus to this again and get the next in sequence right. We're not going to do it because life gets very boring L, L minus 2 but it's just a matter of differentiating. But the thing to pick up is that when we do this next – when we differentiate this, this thing is going to become more complicated because we're going to be doing a derivative of this with respect to theta. So we will get a term that goes like sin to the L minus 2 times cos squared and then differentiating this will get our sin to the L back so we'll get 2 different terms. And then when we differentiate again to get upsi. So this is going to be an amount of – this will – it's going to be an amount of sin L minus 2 times co sign squared. Differentiating this we'll get L minus sin to the L minus 2 and then we get a co sign which goes on to that. And we will also have from differentiating this whatever – yes plus an amount call it B of sin to the L theta. So they'll be two terms and it'll all be times E to the minus L minus 2 phi.

And when we differentiate this again in order to get upsi L L minus 2 it'll get more byzantine because this will generate me an L sin to the L minus 3 times cos cubed. This will get back what we had here and so on and so forth. You get more terms, you get a longer thing coming in front of the exponential. So what do these things actually turning out to be? It turns out that what this is is a normalising constant times PLM of cos theta times E to the minus. Sorry, no... Well, no make that LM. If we just keep going this will turn out to be a normalising constant times the associated Legendre function PLM of cos theta times E to the minus – E to the... Sorry that's an I not a minus isn't it – yes.

This should've been a plus E to the IM phi. So this thing I think you may have met this right in Professor Essler's lectures. This is an associated Legendre function probably derived from solution in series by using Frobenius' method I'm not sure – is that right? But fundamentally this is – fundamentally I don't think this is very helpful knowing this is an associated Legendre function. I think it's much more helpful knowing how to do it this way. The normalisation factors take care of themselves. If we put in these square root animals and we start with this thing correctly normalised. How do we normalise this traditionally? What we do is we say upsi LM is proportional, is equal to some function of radius to be discussed times YLM of theta where this thing the spherical harmonic is a multiple of PLM times E to the I M phi. Normalised so that if you integrate D theta sin theta D phi over the sphere of YLM mod squared you get precisely 1.

So the YLMs are correctly normalised so if you mod square them and the scrape them over the sphere they come to 1. The PLMs have a daft normalisation and that's why I don't think you should bother with PLMs they're just stupid functions. Historically they've been defined in a bad way. The YLMs the things to go on but the YLM is actually one of these functions of cos theta times E to the I M phi. So it has a very simple phi dependence this animal here. And we need to under – so now let's ask. Let's just summarise what we have so these things YLM theta and phi are the wave functions essentially they're the wave functions theta phi LM. They're the wave functions belonging to states of well defined orbital angular momentum. That is to say if in the position representation you apply well – yes LZ to YLM you get M times YLM which is a trivial result because this things goes like E to the IM phi. And if you apply L squared to YLM you get L L plus 1 of YLM.

So if you have an electron here's the nucleus. If you have an electron in orbit around the nucleus it seems reasonable to say – it's reasonable to ask what does the orbit of – what does this system look like what are the wave function of the electron look like if the electron has well defined orbital angular momentum. The answer is that it's wave function is going to be a function of R which will – we'll see, we'll tell you how much it's oscillating in radius as it goes round and round times one of these YLM things. So these YLM things should give us – we should be able to understand them in terms of orbits at some level right.

So let's address ourselves to that. What can we understand about these mathematical functions YLM in terms of what we understand intuitively about how an electron should go in and orbit around it's nucleus. So the place to start is not – is when L is large because when L is large is when we're sort of approaching the classical regime for which we have some grip and the pictures at the top here are – these are contour maps of the real part of YLM. So YLM is an inherently complex thing, right. YLM consists essentially of PLM, some real function of cos theta times E to the IM phi. So by focussing on the real part of that complex function we've got that PLM times cos M phi.

And these ones at the top are all for L equals 15. This is for M equals 15. This is for some intermediate 1 M equals 7 and this is for M equals 2. So what's the physical interpret – what's the physical interpretation of these. This thing, this, this and YLM is a function on the sphere, right. It assigns a complex number to each point on the sphere so this has been – the real part is a real number on the sphere. And what's been plotted here are contours of constant value of this real number on the sphere. So you have to imagine that these are pictures of spheres.

The – so what do we see here is that around the equator we have – so dotted contours mean negative values are the real part and full contours means positive values are the real part. So the large values of the real part are around the equator here and that's apparent from this maths because we know that Y, this is YLL for L equals 15 so in fact it's sin to the 15 theta E to the 15 I5, right that's what this thing here is. And if, sin theta is 1 on the equator and less than 1 everywhere else. If you take a number that's less than 1 and raise it to the 15th power you have quite a small number. So you were expecting that the number gets small quickly as we go away from the equator. That makes – that's exactly what we expect on physical grounds because the state L equals 15, M equals 15 means you've got 15 units of angular momentum broadly speaking and they're all of them parallel to the Z axis.

So this thing is an electron that's orbiting with it's – in a plane, classically it'll be orbiting in a plane. The equatorial plane that was perpendicular to the Z axis. So where do you expect to find the particle, you expect to find the particle in the equator no where else. Where does the wave function peak in amplitude, in the equator and no where else. Why is it segmented like this? Like an orange, right. It's – we have sort of waves going around the equator here it's big, small, big, small, big, small. That makes perfect sense because the change in the – because P the momentum is minus I H bar D by D, D by P position right.

So if you have something with a large momentum it, it's to do with a large gradient. A large rate of change of the wave function. Now the amplitude of the wave function does not change one iota as you go round the equator because this thing has amplitude which is signed to the 15th power of theta. So it's completely constant at 1 round the equator but the phase of this wave function is changing like crazy as you go round the equator because it's E to the 15 I phi. And that is expressing the fact according to this that the momentum of the particle is directed tangentially around thee equator. It's rushing around the equator, it's in the equator and it's rushing around the equator what else would you expect that's exactly what should be the case.

Let's go now to this case which is oops I've lost it. M equals the extreme right one, M equals 2. So we've still got sin, sorry we've still got L equals 15 but we have M equals 2. So we've got a particle which has 15 units of angular momentum but only 2 of them are parallel to the z axis. So classically what this amounts to is that here's our sort of notional sphere and you think that the orbital plane classically would be tilted like this, well even more so sort of like this ish. Very highly inclined so that the spin axis of the orbit was pointing almost in the X, Y plane. So what we're expecting is that the motion is mostly from the northern hemisphere down in the southern hemisphere and back up again. So we expect the contours on which the phase of the wave function changes rapidly – oh fiddlesticks this. It's so annoying.

The direction which the phase varies should be from north to south and lo and behold it is, right. So now we have – instead of having an orange peel plan we have, we have sort of rings going around on which – almost on which the phases. So – and if indeed we had – we put M equal to zero we would have a wave function which had no variation as you went around the sphere it would all be variation as you go from the northern hemisphere to the southern hemisphere which corresponds to the fact that the particle is moving this way.

Now this particle has most of its angular momentum in the X, Y plane but the thing is we don't – because we know, because we know the angular momentum in the Z direction and LZ does not commute with LX we do not know how much angular momentum it has in the X direction. Most of its angular momentum is in the X and Y directions but we don't know whether it's positive or negative. So that means that we cannot in this picture see an orbital plane. The probability of finding the particle is sort of large all the way down here and all the way down there. And if M was zero and the angular momentum vector were exactly in the XY plane. We would have absolutely no variation in the probability to find the particles we went around and around the sphere when in fact even now we have no probability to go round the sphere.

The real part – so what this thing is it's a function of theta times E to the I 2 phi. So the phase is varying as we go around the sphere but in fact the amplitude is not varying as we go around the sphere. The amplitude to find the particle is constant as you go around the sphere on small circles. And that is associated with the fact that we do not know, we're not allowed to know. It is forbidden to us to know which way this angular momentum vector is pointing. But where are we mostly likely to find the particle. Are we likely to find the particle most likely in a given patch on the equator or most likely to find it on the pole. Well this wave function is largest at the poles the North Pole and the South Pole because it's going around – this particle's going around almost – over the poles in a plane which is of unknown orientation.

So we do know – there's great uncertainty. There are many places where it could cost the equator but what we are sure of is it goes close to the pole. So that's why the probability is a sort of crowding of the – imagine a bunch of circles for a polar orbit going round the sphere at different orientations they'd all pass through the pole. There'd be a great crowding of the circles near the pole and that generates the high amplitude to find the particle at the pole a relatively low amplitude to find it at the equator but not a vanishing amplitude to find it at the equator because it does cross the equator twice in each cycle.

So this, this amazingly this sort of – this is an intermediate case M equals 7, L equals 15 this curious mess of squares in which the – you can see the real parties alternately positive and negative the contours are dotted and full. This represents the situation where the orbital plane, in classical physics the orbital plane would be just moderately inclined at 45 degrees or 30 degrees or something to the equator and there is absolutely no orbital plan visible there. And this is where we come to a key point that if you want an orbital plane to be visible and after all the orbital plane of the earth is entirely visible and the earth presumably moves according to these principles too we have to have – how do we get an orbital plane to emerge. The way we get an orbital plane to emerge is by quantum interference between many states that look rather like this and have a patchwork of pluses and minuses.

If you have several of those patchworks of pluses and minuses you can get the amplitude to cancel most places except in some inclined orbital plane. So it's uncertainty in the angular momentum which will generate for you if you want it some degree of certainty in the location of the orbit going around the sphere. It's the old uncertainty principle over again. So those are the classical – this is almost the classical regime up here – right. Of course as the earth goes round the sun it's angular momentum is who knows 10 to the 50 H bar or something right it's simply I haven't worked it out it's some staggering number. So you would have to imagine 10 to the 50 little patches here of pluses and minuses or may be it's 10 to the 50 squared, I think it probably is 10 to the 100 patches of pluses and minuses. And then you can by taking a number of those may be you take 10 to the 34 of those with 10 to the 50 patches you'll be arrange for exquisite the pixels to cancel everywhere except in some extremely narrow band which is the `[[?? 0:27:12]]` orbital plane of the earth.

So atoms don't live in that regime up there of L equals 15. Atoms live in this regime, this tiresome regime down here. This is – where am I? This is L equals – I've lost it. This is L equals 1 and these are the three things for L equals 2. So this is Y11. So that means you've got 1 unit of angular momentum and – well it doesn't actually right because what does L equals 1 mean? L equals 1 means that L squared has answer 1 1+1 equals 2 so the total angular momentum the square root of L squared has answer root 2 which is distinctly bigger than 1.

So we've got as much angular momentum along the Z axis in this 11 case as we can which means the particle definitely is going around the equator so – and you can see that it's going around the equator well I can't from this angle but I hope you can in the sense that the thing isn't constant. The wave function has gradient as you go around the equator there's a gradient. On the other hand there is not a very high probability of finding it in the – this is only the real part of the wave function if we would look at the imaginary part of the wave function well how does this one – this one goes like sin theta, not like sin to the 50 theta this function here is sin theta times E to the I phi.

So as you – in the equatorial as you go away from the equatorial plane the amplitude to find the particle falls but only falls like sin theta so it's really quite likely not to be an equatorial plane and that's associated with the fact that although we've done our best to get the angular momentum along the Z axis it isn't along the Z axis because it's total angular momentum is 1.4 something times H bar. And only one of those units is along the Z axis so it's some sense inevitably inclined and this is the case when we have no angular momentum along the Z axis so this is the case of polar orbits the amplitude to find the particle is greatest at the two poles, smallest at the equator etc etc etc.

But the whole picture's less clear cut. And I won't bore you by talking about these but it's worth thinking about the L equals 2 case to see what extent you can make sense of these – physical sense of these pictures here.

Okay so now we should address an important topic which is the parity. This is practically an important topic the parity of YLM. So remember the parity operator P working on upsi makes a state whose amplitude to be at X is minus – is the amplitude to be at minus X if you were in the state upsi. That's the definition of the parity operator and these states of well defined angular momentum turns out have well defined parity that's what we're about to show and what's more the parity is minus 1 to the L. So states of different angular momentum have alternating parity some are even parity some are odd parity that's what we want to show.

Okay so…as what we do now is… So this is sort of imagined in cartesian coordinates we need since our Ys are all defined in terms of polar coordinates we need to translate the operation of going from X to minus X into spherical polar coordinates. So as X – as we go from X to minus X it's easy to check that what happens is that theta goes to pi minus theta and phi goes to phi plus pi. So this reflection action you need a picture really well we can just about show it I suppose I hate three dimensional pictures because the three dimensional picture okay. Here's theta the spherical polar coordinate theta and what you do – what we have to do is take this point and move it down here, right. And what we do is we move this point down to here that's the theta so theta, this theta goes to pi minus theta and then having got it down here we rotate it through out of the board and back into the board through pi and phi and that's how we get it down here. So these are the changes in polar coordinates that are associated with that.

Now YLL – oh yes well what else can we say when – if – so theta goes to pi minus theta what does that have to say about sin theta. Sin theta goes to sin pi minus theta and sin pi minus theta it's easy to check for a variety of arguments is actually equal to sin theta. So sin theta doesn't change and E to the I L phi what happens if you add pi to E to the I L phi well you're adding E to the I, you're getting an extra factor E to the IL pi which is minus 1 to the L times E to the IL phi – is that right?

Okay now YLL is a constant rather a yucky constant so I won't bore you with it times sin – we've proved this sin to the L theta E to the I L phi. So this thing does not change sin or it doesn't change at all, right. So we can say now that YLL goes to – this doesn't change sin, it doesn't change at all. And this one changes sin. So it goes to minus 1 to the L of YLL. That's under X goes to minus X so that this means the parity of YLL is even if L is even and odd otherwise. That's a very important result and moreover it generalises because we have that YL L minus 1 is L minus over some square root that's really boring, well it turned out to be 2L so may as well put it in times YLL. And what about this? What's L minus, L minus is LX minus I LY in the position representation what is this, this is minus I H bar of Y D by DZ minus D by D Y plus minus who knows H bar. It doesn't much matter. The key thing is that we're going to have here is a Z D by D X minus X D by D Z.

And when we change X to minus X, Y to minus Y and Z to minus Z these things we get change of sin here and a change of sin here, a change of sin here, change of sin there so L minus and also as a matter of fact L plus is unchanged by P. The strict mathematical statement is that the parity operator commutes with either of these animals. Indeed all the angular momentum operators commute with the parity operator basically because they contain products of positions or if you like ratios of positions, whatever, they don't change. So what that means is that this is going to have the same parity as this because if you apply the parity operator to this you're applying the parity operator to this, those can swap in order. This turns to minus itself, the minus sign can be taken out and therefore we've shown that that leads to the conclusion that this thing has the same parity as this. Let me just write that argument down perhaps. So we have that P L minus, sorry P on YL L minus 1 is equal to P L minus upsi, sorry not upsi, Y LL over some square root that's not interesting is equal to L minus PY LL over the square root which is equal to minus 1 to the L times P times, sorry this thing produces YLL. So we have L minus YLL over the square root but this is YL L minus 1 so it's equal to minus 1 to the L of Y L L minus 1.

So we conclude that Y LL, LM has parity minus 1 to the L for all M. This is a very important fact because it enables you to set to zero all sorts of integrals which would otherwise be very tiresome to work out. How we doing? Yes I've just realised that there's one other thing so which we've unfortunately lost – is it coming back? No.

What I wanted to do was show you the forms of the YLMs. The first few you need to have some sense of how they go right. So Y nothing, nothing is 1 over the square root of 4 pi. Y which order they put in yes so we want Y 1 nothing is basically cos theta there happens to be some factor of root 3 over 4 pi. Y 11 is of course sin theta times some normalising factor E to the I phi. The Y1 minus 1 would be the same thing with the minus sign here. So the point is that the Y1s go like cos theta and sin theta and the Y2s go like cos theta sin theta so Y2 nothing is equal to a normalising factor that happens to be 5 over 16 pi that's not so interesting times 3 cos squared theta minus 1. And Y21 is minus the square root of 15 over 32 pi which is not so interesting what's important is it goes like sin 2 theta which can also be written as sin theta cos theta and the other one goes like, of course, sin squared theta – this one has E to the 2I, sorry E to the I phi and this has E to the 2I phi.

So what do we need to remember. What we need to remember is that obviously Y00 has no angular dependence. Yes okay so I think the machine has finally come back to life and the correct formulae are here that's what I wanted to show you. The Y1s have a cos or a sin, the Y2s have a Wif, well you can either think of – they have a strong Wif of cos 2 theta and sin 2 theta, right. Because cos squared theta is something like a half of 1 plus cos 2 theta so there's a Wif of, this could be rearranged to involve cos 2 theta. Here we have a sin 2 theta and this sin squared has a wif of cos 2 theta about it because we know that sin squared theta is a half of 1 minus cos squared – of cos 2 theta, something like that, right.

So we have these double angle formulae and so it would go on. If we were looking at Y3 we'd have these things would have dependencies that looked like cos cube, cos 3 theta and sin 3 theta right. That's the pattern. But you don't need to know about the pattern beyond here but these patterns you're expected to be able to sense so that when you're given a function of theta which is made up a linear combination of these things you need to be able to unscramble it and write it as the right linear combination of those Ys.

Right, the next topic. So in preparation for work on atoms we need to get an important formula for how kinetic energy can be expressed in terms of L squared. And this finally obliges us to face up to the tedium of calculating what L squared is, what differential operator represents L squared in the position representation, right. So we start by observing that L squared is – it can be written as L which way round do I want to write it? Yes plus/minus. Okay I want to write it – I can write it either way but I meant do it consistently like this, well let us – let's see what we're going to have to add to this to make L squared. This is LX plus I LY LX minus I LY – what's that going to come to? That's going to come to LX squared plus LY squared plus well minus I LX, LY commutator. That's what this thing multiplies up to. If we want to get L squared we'd better – here is a good start on L squared, let's add LZ squared but we need to get rid of this LX, LY is ILZ so we've got here what with this minus sign a plus LZ. We've better take an LZ away in order to square the books. So that's what this should be. Sorry this should be put equal to plus LZ squared minus LZ. So that's that.

So what we do now is we write down L plus L minus which we have floating up there in the stratosphere so we have L squared is equal to E to the I phi D by D theta plus I cotangent theta D by D phi. And that should operate on L minus which is minus – I'll take the minus inside the bracket E to the minus I phi D by D phi – D theta sorry. This minus sign was up there outside the bracket I think plus because I propagated the minus inside the bracket I cotangent theta D by D phi. So this disgusting mess is that product and then we have to add LZ squared and take away LZ.

This thing is minus, LZ is minus I DBD phi so with that minus sign we get a plus I D by D phi and this is going to be minus D2 by D phi squared. So the name of the game is to differentiate out this piggy mess and find out what it simplifies to. Some parts of it are easy, right. We're going to have for example – at the end of the day we will have terms where this is multiplying this and this is multiplying this and these two exponentials have killed each other off. So we will have a term like D2 by D theta squared these Is will generate – sorry there'll be a minus D2 by D theta minus because one of these has got a minus sign. These two will create me a minus cot squared D2 by D phi squared. That's the easy part.

Right, now the mess. There's going to be some mess because this differential operator is going to bang into that okay. And generate a minus I times what will kill this off so we'll have a minus I – oops oh no but then it's times this so the minus I that we're getting from here will meet this and generate a plus 1. So we have cotangent theta that's this cotangent theta times this bracket. So that's the result of this differential operator seeing this. When this differential operator sees this bracket or we get – oh actually sorry we get a mixed derivative term we get two terms. We get one term that we've already written down and we get a term D2 by D theta, D2 by D phi but that is going to be cancelled by a term that comes from here when this differential operator looks at that. We'll deal with the differentiation of this in a moment. So I'm not going to write down those mixed derivative terms.

Otherwise we have – we've now – so on that understanding we have dealt with the action of this on that. Now what about this one. We've got the operation of this on this. We've got the operation of this on this I've just said that that's cancelled away. What we haven't got is this. When that differential operator meets this we get the differential of cot is cussock squared I think so I think what we have is plus I cussock squared theta of D by D phi.

Now the sin should be checked at this point because sin's are a pain right. Well I think it must be that the derivative is minus cussock squared, votes are going to be taken afterwards. Right so that's the derivative of this on this and then I claim that these brackets are dealt with and all we have to do is write down the trailing terms here which is a plus I DBD phi and a minus D 2 by D phi squared.

Now we need to consolidate our various terms. We have three terms one, two, three which are just D by D phi terms and god be praised they all add up to nothing because we have a trig identity which is cot squared minus cussock squared is minus 1. So we have that cot squared theta minus cussock squared theta is minus 1 and here's our cot squared. There's our cussock squared and I'm missing – and this should've had a – I, I, I, I, I we have an I problem, right.

These have to be all – no, no, no I'm not trying to mess with that one. I'm not trying to mess with that. Right I'm going to have a cot squared here with associated attendant I. I've got a cussock squared with an I and I have here a 1. So let us buy that that causes those all to add up to nothing. Then I also can use this identity to consolidate this double derivative and this double derivative. So we have a cot squared and a 1 and I can trade it in for a cussock squared according to that formula there, right.

So we end up with minus D2 by D theta squared it's going to be cot squared minus cot squared so it's going to be – you've got cot squared of the thing they both carry minus signs which means they have to have them on the other side so we get a minus cussock squared according to this. I'm slightly worried about this. So I'm going to end up with a cussock squared D2 by D phi squared. And I strongly suspect that sign is wrong but that's what I've honestly got. So that's this dealt with and the only – this is – so this has been dealt with, this has been dealt with, this has been dealt with. I think we're all tickety boo. We're not are we? What have I lost?

To believe this is the easiest way to do it it's hard to believe isn't it but it is, it is.

Male The single derivative of D by D theta?

Contributor This one.

Male Yes.

Contributor Right. So that remains cot theta D by D theta, thank you, right. So we now consolidate this all being well into 1 over, minus 1 over sin theta D by D theta sin theta D by D theta. And this should be I think a minus, that sign is wrong. A 1 over sin squared theta this is how it's usually written D 2 by D phi squared. So I've screwed up on the sin there somehow.

So when you differentiate this we get a cos which cos over sin is cot so that's this term here. We have the double derivative sin etc etc etc. And what is this? This is R squared times the angular part of Del squared. And on that note it's time to leave. We're not quite finished with the calculation but that's the important bottom line that L squared is actually with a minus sign, minus R squared times the angular part of Del squared.

And we'll push that forward into the kinetic energy tomorrow, no on Wednesday.
