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Contributor Okay, so we finished last time, we just about pushed through the calculation of what L2 is as a differential operator, which we did, if you recall, by multiplying L, the ladder operators L minus and L plus together. It was rather a tedious calculation, but at the end of the day, with luck, we ended up with this. And we should recognise that this, L2 is minus this combination of partial derivatives with respect to Theta and Phi, is the. It's minus R2 times the angular part of Del2 the Laplacian when looked into, when put into spherical polar co-ordinates. So if you take this thing and put it here, this minus sign cancels that minus sign.

And we get one over R2 sine Theta D by the R Theta, sine Theta etc., which I hope you recognise as Del2. So you might ask yourself, so physically what's happening? We have the kinetic energy operator, i.e. which I've put this as H sub K which means P2 over two M, where this is PX2 plus PY2 plus PZ2 is also minus H bar2 over two M times Del2. So in the position representation, this operator becomes this, right. Because P is minus IH bar times gradient. And classically, we have that L is equal to V tangential times the radius.

So L2 is V tangential2 times radius2. L2 over R2 is equal to V tangential2. So we have HK is equal to, well, sorry, that's suggested something hasn't it? That this nabla squared can be written in terms of some radial derivative plus, so we could say that HK is equal to some radial minus H bar2 over two M, one over R2 D by the R, R2 D by the R. And then we're going to have plus H bar2, L2 over R2, I think. Just by substituting into there, over two M, sorry. And what's this going to be? We defined the angular momentum operator L2 to be dimensionless. So putting an H bar in front of it, H bar L, is the classical animal, right?

So H bar operator is the analogue, I should say is equal to, it's the analogue of classical angular momentum, total angular momentum. So this that you will have here, as the dimensions of total angular momentum squared, it's the classically understood thing. So this term here is looking awfully like V tantential2, sorry, I need a mass here. Right, the classical angular momentum is M V tangential R. So the square is M2 V tangential2 R2. Move the R2 down here and this is the classical relationship that L2 over R2 is M2 V tangential2. So this is looking like, this in the back here, is looking like half M V tangential2.

That's what this suggests to us, it's a quantum mechanical formula which is correct, but it's suggesting to us that it's this, the sort of natural translation of classical physics is this. And this is clearly the tangential kinetic energy. So this is the K bit, the tangential kinetic energy associated with tangential motion. Which suggests that this here should be the kinetic energy associated with radial motion. And that's what we want now, to put on a firm intellectual footing. So we're going to show that this thing is minus H bar2 over two M times PR2, where PR is the radial momentum.

So the question I want to address now is, what is the radial momentum operator? We found the tangential operator. We found it in some sense the tangential momentum operator in this sense L and now we want to find the radial one. So classically, momentum is a vector and we can say that the radial momentum is simply R dot P over R. In other words the unit vector R dotted into P must surely be radial momentum, momentum in the radial direction. But there's a problem with this from the perspective of quantum mechanics, because this operator doesn't commute with this operator.

So it's, well, what does that mean? This thing, in QM R dot P over R is not emission. Let me prove that to you by, it's easier to prove that in general than in particular. Okay, so let me add two emission operators, A dagger is A and B dagger is B. Then let's look at AB dagger. If I'm multiplying these together, what is, I get an operator AB. Is this operator emission? Find out. That's B dagger A dagger because the rule for taking emission adjoint is you reverse the order and dagger the individual bits. But B is B dagger and A is A dagger, so this is equal to BA. So is this equal to AB? Well, clearly it is if, and only if B and A commute.

So this is only if A,B equals nought. In words, the product of two of the emission operators is itself emission, only if those two operators commute. If and only if those two operators commute. So this, R dot P, which is shorthand of course, for XPX plus YPY plus ZPZ, would be emission, could be emission only if X and PX commuted, Y and PY commuted. Well they don't. Therefore this is not emission, therefore this is not an observable. So it can't be what we're looking for. We're looking for something, the momentum in the radial direction which is observable-ish.

All right. Well there's a fix to this problem, there's a general fix and we're going to use it, but if we do a half of AB plus BA, this is emission. Because if you take the dagger of this, this one, we've just proved the dagger of this one is that and the dagger of that one is this. So this thing, the dagger of this bracket is itself. So when you've got two non emission operators, sorry, you've got two emission operators that don't commute, and you want to make the product, the way to go is to take the average of them, you know, it's a really naïve thing to do. So let's do that.

So we try, let's have a look at the emission operator PR which we're going to define to be a half of R dot P over R, where it's important that that R, this R here, is in front of the PR plus P dot R over R. So this thing here will be emission and I'm going to show that it is what we require. So in the position representation, so you can do this calculation in the abstract, not in the position representation, but it's easiest in the position representation, so that's how we'll do it. So PR is equal to, so this P gets replaced by minus IH bar grad, right? So this is going to be minus I H bar over two common factor.

We're going to have R dot grad over R, this of course is the scaler R. plus the divergence of R over R. Now the issue is this. When, this isn't quite the divergence of this. What this means is, this is remember an operator, it's waiting for a wave function to come and stand in front, to get operated on, right. So this differential operator operates on everything to its right. It operates on this and it operates on these two. This here, this differential operator operates on everything to its right, which is only the upsi. So we have to, when we expand this out, we then get three terms.

Because we're going to get thing operating on this, that and that standing idly by. This thing operating on this, with this and this standing idly by. And this thing operating on this with these two standing idly by, which is the same as that. So this is going to be minus I H bar, R dot grad over R. So I'm taking this one that I've got and the one that I'm promised at the end of all this reduction here. So that's where the two went to. So that's those two. And now I've got these two bits, minus I H bar brackets. We're going to have, over two, sorry, that's this factor here. Then I'm going to have this thing operating on this, the divergence of R is three.

And then I've got this thing operating on that and it's going to be R dotted into the gradient of one over R. So that's going to be minus R dotted into the gradient of one over R. The gradient of one over R has to be, well, it is the vector R over R2. This minus sign comes from the differentiating of the one over R, right? Because I'm reminding you of previous maths now, the gradient of R itself is the vector R divided by R. This is a dimensionless animal, because that has dimensions of one over length, that has dimensions of length. So it's the vector, it's the unit vector R and that's what we've been using. Whoops, sorry I've made a mistake, this should be R acute

Just to fill in here, so the gradient of one over R is equal by the ordinary rules of differentiation minus one over R2 times the gradient of R. But the gradient of R I've just said, is vector R divided by R hence the R acute. These two dock together and make an R2 which cancel most of those, so this minus sign, these two can be combined to a two over R, the twos go away and guess what we end up with? It's minus IH bar R dot grad over R plus one over R. So that's what this stuff reduces to. What we next want to know is so what is R dot gradient? Well, I want to know what this is in spherical polar co-ordinates.

Well, the thing to do is just to write down, let's write down RD by the R. It's easy to see that that is going to be X, well let's do it. R then we're going to have by the chain rule, the X by the R, D by the X plus the Y by the R, D by the Y plus DZ by the R, D by DZ. That's just the chain rule. But what is the X by the R? X is equal to R sine theta cos phi, so the X by the R is equal to X over R, for the same reason DY by the R is equal to Y over R and so on and so forth. So this is equal to X D by the X plus Y D by DY plus Z, D by DZ. Because this is X over R, but this R cancels the R on the bottom. Y over R, R cancels what's on the bottom.

And what's this? This is a vector product of X,Y,Z with nabla, D by DX, D by DY, D by DZ. In other words, this is the animal that we're interested in, R dot grad. So I have now that PR is equal to minus I H bar. R dot grad we've just agreed is R D by the R, so those R's go away and we have D by the R plus one over R. So we have an interesting result, we have that the momentum associated with the radius is not simply D by D radius like the momentum associated with X is D by DX. There's also this additional term in here. But just to convince you this really is the momentum associated with radius, let's for fun, calculate R, PR.

So what is that? That is minus I H bar of R D by the R plus one, which is R times PR minus D by the R of R minus one. Sorry, I'm forgetting zero aren't I, that's the trouble. I want to get I H bar out of this, what the hell did I do wrong? D by DR, PR minus D by DR plus one over R working on R. Yes, yes, yes, sorry, yes, that's perfectly correct. Okay, so the ones go away, this sort of thing is confusing. Right, now as I say, what does this mean? This means D by DR of everything to its right and there's a phantom wave function here waiting to be operated on. So this is the derivative of R apsis.

When we take the derivative of the R, we get one times apsis and then the R stands idly by and we do the gradient of apsis. The second term cancels on this, because R times the gradient of apsis is occurring here with a plus sign and there, it will be occurring with a minus sign. So what we're left with is the D by DR times apsis. The D by DR which makes one, so this is equal to plus, because it's a minus sign coming here, I H bar. So that it's these two operators satisfy the conomical commutation relations, right. Conomical commutation relation. So PR really is the momentum associated with R.

Okay so what are we really trying to do here? We're trying to show that that one over R2 D by DR of R2 D by DR is essentially PR2. So let's calculate PR2. PR2 is going to be minus H bar2 because there will be two minus IH bars. And then it's D by the R plus one over R brackets D by the R plus one over R, which is equal to minus H bar2. Obviously this on this is D two by DR2. We will get, this differential operator will differentiate that and produce a minus one over R2. We will otherwise get a one over R D by DR and also a one over R D by DR. So we'll get two of one over R D by DR. And, sorry, and I haven't finished. I also get this thing on this thing, is a plus one over R2.

So these two terms cancel and we're left staring at this. I should have had two of these terms, I think I said I was going to get two terms, because I have a one over R D by DR and I have a one over R D by DR. After this operator, when this operator works on this, it produces that, but also it works on the phantom wave function sitting over here, with that standing idly by. So we get two of these. I think I said that, but I didn't write it, I'm not sure. So we have a minus H bar2 D two by the R2 plus two over R, D by DR which can also be written as minus H bar2 over R2 D by DR of R2 D by DR.

Because if you differentiate out this product, you get R2 on R2 times D two by the R2 which is this term. And you also get a two R over R2, two over R times D by DR. So here, this term here, we've now shown that HK, the momentum operator, which is minus H bar2 over two M Del2 is also minus H bar2 over two M of, sorry, yep. Well, let's leave that outside. Let's take the H bar2 into the bracket. We're going to have a one over R2 D by DR, all that stuff, which we've just shown is PR2. And then oops, there was a minus sign, so that soaks up this minus sign of PR2 and then similarly there's plus H bar2 L2 over R2.

This is a very important formula that we will need when doing hydrogen and therefore fundamental to. So it's expressing your kinetic energy in terms of your radial kinetic energy and your tangential kinetic energy. And that's one of the reasons why the total orbital angular momentum operator is important because it encodes your sort of energy going round and around. So with that, we are now finished with, we can mercifully finish with orbital angular momentum and we can get onto spin.

This is somewhat more interesting in the sense that it's, quantum mechanics has more remarkable things to say and it's less tedious, because all that stuff with those partial differential operators D by D theta and stuff is not much fun, it has to be said. Right, so we have identified two types of generated rotations. The total angular momentum operators and they generate, we introduce them in order to generate complete rotations. So U of alpha being E to the minus I alpha dot J rotates system as on turntable. So it moves your system around the origin.

It's as if you put your system on a turntable, centred at the origin, with its axis at the origin and you turn the turntable round. Your system moves through space and it rotates simultaneously, whatever internal structure it has. But we also have shown that LI, the orbital angular momentum, moves system on circles. So it moves it around, physically it translates it around the origin, but it does not rotate it at the same time, it leaves its orientation fixed. And we have some, well okay, so we've found the commutation relations here. We've found that JI,JJ is equal to I sum over K epsilon I J K J K.

And we found that it was also true that LI LJ was equal to I summed over K epsilon I J K L K. They have the same commutation relations amongst themselves these operators, which is why we could use the work we did demonstrating what the Eigen values of these could be. Also down here, this implied that J2 has I values J J plus one for J is nothing, a half, one, three halves etc. And from these commutation relations, we inferred that those are possible values for the Eigen values of these operators, but we also had the principle that if we translated something completely around the origin, we proved that that was the identity transformation.

So we concluded that L L plus one had to be L equals L equals nought, one, two, integers only allowed in this case. What we're now going to do, is introduce SI, is by definition JI minus LI. It's the difference between these two. What does that mean physically? It means that SI is going to be the generator of rotations of a thing about its own axis. So we're not going to be. This rotated on a turntable, so it rotates it and moves it. This simply moves it round a circle. So this is going to only rotate it on its own axis. It's not going to move it. It's only going to rotate it.

That's what we expect to happen, but we'll have to be guided to some extent by the mathematics and what is the mathematics? So having introduced these newfangled operators, it's important to figure out what the commutation relations are going to be. Now SI,SJ is going to be JI minus

LI,JJ minus LJ. We're going to get this commuting with this. So we're going to get I epsilon summed over K, epsilon I J K. This commuting with this will produce the JK. This commuting with this will produce an LK. This commuting with this. Now, we didn't write that down, but JI commuting with LK, this is a vector operator and therefore this thing, when it commutes with this, always produces the missing component of this vector.

So this is going to be minus LK and similarly, this thing on this thing is going to produce, swap them over and you're going to, well, there are several signs here that we could hound down. But this thing, we're looking fundamentally at the same thing, it's the commutator of this on this. We're looking at minus LI,JJ is equal to obviously JJ LI is equal to I epsilon J I K L K is equal to minus I epsilon, that's summed over K this is. Sum over K of epsilon I J K L K. So J I K and, but I would like to add this in the order I J K, so I swap those two over and introduce the minus sign to compensate.

And then you can see that this thing including that minus sign, is the same thing as this thing including that minus sign. So we have a minus another LK. So this is the justification for that last term there. So what do we end up with at the end of the day? These three LKs collapse into just one LK. It's going to be JK minus LK, in other words this is going to be I summed over K epsilon I J K S K. So these spin operators. Sorry, did I say that they're going to be, we call them the spin operators. They have exactly the same commutation relations as the J. Therefore we know what their Eigen values are.

So this implies that the Eigen values of S2 which is, of course, is SX2 plus SY2 plus SZ2. R S S plus one where S is equal to a half, sorry nothing, a half, one, three halves blah, blah, blah. Okay, because these results follow merely from the fact of having the commutation relations SI,SJ is I epsilon I J K S K. Are the half integer values allowed? Answer, you will have a half integer when J does, because L does not. Right, why is that? That's because SZ is equal to JZ minus LZ and JZ,SZ equals nothing, which is also the same as LZ,JZ etc. All these three operators commute with each other, so there's a complete set of mutual Eigen states.

So we can now see that if this has half integer, this is using half integer Eigen values. So the Eigen values of this, are going to be the difference between the Eigen values of this and the Eigen values of this. So this has half integer Eigen values, therefore this will have to have half integer Eigen values, because this one has integer Eigen values. So if J has half integer Eigen values, then S does. Correspondingly, if J doesn't, S doesn't. It just tags along behind J. And indeed that's how we tend to think about it. We tend to think that the integer amounts of angular momentum come from orbital motion LZ.

And the half integer values, if present, come from SZ and that's why J has half integer values. That's how we tend to think about it. And I think I have claimed a few times that spin is something to do with the orientation of our system and now it's time to make good this claim that the Eigen values of the spin operator or your response to the spin operators encodes how a particle is oriented. And this is a strange area, a very quantum mechanical area. Okay, so in general, the internal configuration of a system could be written, we could write upsi is equal to the sum of S M upsi S M. We've got a complete set of mutual Eigen states of S2 and SZ.

So we're saying that S2 on SM is equal to S, S plus one of SM. And we're saying that SZ on SM is equal to M SM. And there should be a complete set of Eigen states of this, mutual Eigen kets of these operators. So I should be able to expand...