Title	*023 Spin 1/2 , Stern - Gerlach Experiment and Spin 1*
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Contributor Okay so let's get underway. We were talking about the `[[spin a 0:00:07]]` half, the most important type of spin yesterday and we got this far. So any state as regards its spin, its orientation should be expandable as a linear combination of the state plus which means you are certain to get a plus a half if you measure the spin along the z axis and minus. And there will be some coefficients, there will be these coefficients here and a complex number here and a complex number there.

The amplitude to measure plus a half on sz or the amplitude to measure minus a half on sz and we're calling these, it's obviously handier notation to call that thing a and this thing b. And then what we want to be able to do is write the result of using some spin operator on this arbitrary state upsi, we call that phi. We can also expand as a linear combination of this and this because they're a complete set of states for the orientation for this spin a half particle, spin a half system.

And we I hope I persuaded you yesterday that these numbers, these amplitudes c and d can be obtained as the vector on the left. If on the right we put in the two numbers that characterise upsi on the right we get out on the left the two numbers that characterise phi after we've multiplied by this matrix of four complex numbers being the expectation value of the relevant - of whatever operator we're trying to use between the plus states, the minus states and then these non-classical off diagonal bits on each side.

And we said I think we finished by saying that if iz in other words if we're interested in the result of using sz on upsi then this matrix is very simple because sz on plus is simply a half of plus. So we get a half appearing here, we get minus a half appearing here because sz of minus is minus a half times a minus. And we get nothing appearing here and here because plus and minus are orthonormal. So we have this diagonal matrix which is no accident. It is simply the matrix that contains the eigenvalues of sz down its diagonal because we used as basis vectors the eigenkets of sz.

We made that choice and the result is that s the matrix representing sz is diagonal with its eigenvalues down the diagonal. And this matrix is conventionally written as a half times this matrix which is called sigma z. And is called a Pauli matrix because Wolfgang Pauli introduced it into physics although it was known to mathematicians, matrices like this.

Okay so more interesting is if we ask ourselves what's the matrix for sx? So the matrix for sx is going to involve things well we're going to have for example plus sx plus. This is a complex number we want to know which complex number. And the secret of calculating this is to write sx as a half of s plus plus s minus where s plus minus of the matrices, sorry the operators, that we already introduced in the context of j and l to reorient the angular momentum either towards the z axis or away from the z axis.

So as these are sx plus and minus i times sy right? So this operator was introduced in the form of j plus minus but remember spin and total angular momentum have the same commutation relations, the same behaviour in every way. So these ladder operators are this and obviously if you add s plus to s minus you got 2sx because the sy terms cancel. So this is definitely the case. So this thing here can be written has a half of plus s plus plus plus plus, sorry sorry oh yeah that's what I'm trying to calculate yeah, s plus s minus plus.

S plus tries to raise this to an even larger value. This is plus a half. It will try and raise it to plus three halves but no such value is allowed because of spin, the total spin is only a half. So it kills it in the process therefore this one is zero, s minus successfully lowers this to minus but minus is orthogonal to plus so this is zero. So this element here is zero and that's the top left corner of the matrix for sx is zero.

Similarly exactly the same reasoning would lead you to conclude that the bottom right hand corner is zero and the non zero elements occur off diagonal. So if we look at plus sx minus we're looking at a half of plus s plus minus plus plus plus s minus minus. S plus raises minus to plus successfully. S plus on minus is exactly 1 times plus. So this number here is equal to 1 and minus tries to lower this and kills it in the process. And therefore this is equal to zero. So this element, this off diagonal element is in fact equal to a half.

We know that the bottom right hand element is the complex conjugate of the top right hand element because s is an Hermitian operator. So we know now that the matrix is sx is represented by the matrix half of nothing 110 also known as a half of sigma x the Pauli's matrix. This is the Pauli matrix sigma x. And when we do the same thing to find out what sy is we write this as a half of plus sorry 1 over 2i of s plus minus s minus right? Because if you take the difference of sx plus iy and sx minus iy you will end up with 2isy.

So we have this and what do we get? This s plus raises this minus to plus. So plus s plus minus again equals 1. So therefore this is equal to 1 over 2i also know as minus a half minus i over 2. So the matrix representing sy is going to be a half of 1 minus ii, sorry you only need one, nothing, nothing. The diagonal elements will be nothing for the same reason that they were with x also known as a half of Pauli's matrix sigma y. So that's where the Pauli matrices come from. They're simply the matrix representations of the spin operators in a basis, when you choose as your basis the eigenvectors, the eigenkets of sigma z.

So let's use this apparatus to do something slightly interesting. It's an excellent exercise both in practising getting experimental predictions out of this abstract apparatus. And also we learn something interesting about how the orientation of atomic scale things behave. Somewhat counter intuitive arrangements. Okay. I don't think this computer this system projector system is going to work today for some reason so okay.

So the point is that so the point is that the spinning charged body is a magnetic dipole. I think that's kind of plausible. So electrons neutrons protons etc sorry not neutrons electrons protons being spinning charged bodies have little magnetic moments. They are little magnets. So if you put a magnet in a b field you have this is the energy of a magnetic dipole in a mag field. So there is a minus sign here which says that the energy is lowest when the magnetic when the dipole is aligned with the magnetic field, right?

So when this dot product is positive the energy is lowest. So that's why magnets, compass needles whatever align with the magnetic field. That also means that if a magnetic dipole is aligned with the field its energy will drop as it moves into a region of bigger fields because this will become a more negative number. Whereas if it's anti-aligned with a magnetic field then its energy will increase if it moves into the magnetic field because this will be negative. And the two minuses will cancel. We have a more positive energy.

So since things tend to move in the direction that minimises their potential energy we have that magnets aligned with b will be sucked into a region of stronger b. So a magnet, a dipole aligned with b, so that means that mu.b greater than nought is sucked into a field. So if the field strength varies spatially which it often does particles which have their fields, their dipoles aligned will be sucked into b. And similarly the other ones will be repelled. So the anti-aligned dipoles will be repelled from a region of high b.

So that was the physics that Stern and Gerlach exploited in 1922 in experiments which astounded the world. They found themselves, they made themselves a magnet shall we call this North and we'll call this South. So they made themselves a magnet which had pole pieces, one of which was pointy and the other of which was flat or even well I think it was flat. But it could also be concave like this and then you can imagine how the field lines run. The field lines run like this somehow.

I'm not doing a very good job of it. My diagrams are usually rather rubbish. So the point is that here we have a crowding of field lines which means we have high b near knife edge. So I have a nice picture of this but the computer isn't willing to show it because this is the end view of a long thing. So this is like the point of a knife right? We're looking end on at the point of a knife and this is just a table somehow.

So if you have some particles with some spin coming in here and aim it right so that they're heading for this, well they're heading a bit below this region of high magnetic field like this. Then the ones that have their spin aligned this way into b are going to be sucked into the region drawn, attracted by the region of high b near the point of the knife and move on up here. So these are the particles which have mu.b greater than nought and particles anti-aligned with mu.b less than nought will come down here.

Of course this is all grotesquely exaggerated. In fact you'll have a very subtle curvature and then you'll have a straight line. So we get the particles deflected either way. So what they did was they took silver atoms because silver atoms turn out to be spin a half particles coming in here. Then and they found which surprised them and everybody else that half of their particles, half of their silver atoms went off this way and half of their silver atoms went off that way. So that when they detected their atoms on a screen over here they got two blobs distinctly separated.

The quantum mechanical interpretation of this is that when these atoms are in here they, sorry I haven't said, that mu the magnetic moment is equal to some number, the jar of magnetic ratio times the spin operator. So when they're in here the magnetic field is as it were measuring their component of spin in the direction of the magnetic field. That's what you say to yourself and there are only two answers possible. Either you'll get plus a half or you'll get minus a half for the value of this and therefore mu will be either a half g in the direction of b or it will be minus a half g in the direction of b.

And the half for which it's plus a half of g will be deflected that way and the other lot will be deflected down here. And there you go. So at the end of the day you have a Stern Gerlach filter. You put in the particles they've just come out of some oven. You've heated up some silver in an oven, made some silver vapour, allowed it to diffuse out of some holes, `[[columnating 0:16:46]]` slits and that kind of stuff. So it's coming along here with some thermal velocities.

And out of your filter you have a load of you have atoms which have their spins in this case up on z and the ones that come out here are in this state. So it's a machine for, it's a practical device for creating silver atoms which are in this state. Now you can play some entertaining games by installing another Stern Gerlach filter. So let's just block those off; stop them from being a nuisance. Stick in another Stern Gerlach filter here and now let's measure the, let's measure s sub n.

So let's measure the spin along some unit vector n and let's take, so we're going to have this to be the x direction. We're going to have this to be the z direction. The y direction will have to be out of the board, alright? And what we are going to do is we're going to take n is equal to nothing, sine theta, cos theta. So n is going to be a vector which if theta is nothing is just in the z direction and if theta is phi by two it's in the y direction and it can be allowed to scan between these directions as we vary theta.

And what we want to do is calculate which fraction of the atoms will survive; will get through the second filter. So this is the filter f1 and this is the filter f2 and you want to calculate the probability that an atom gets through both filters. Or let's focus for the moment on the probability that an atom has got through the first filter, gets through the second filter. So the probability that you pass f2 given that you passed f1 in quantum mechanical language is plus a half on n given that, well we'll just say plus on n given that you were plus on z.

So this is the sate that you're in. Up there it's just called plus. When I put in a z to distinguish it from this which is in the direction of n, that this is an eigenket of sz with eigenvalue a half. This is an eigenket of s sub n with eigenvalue a half. This pair of things makes me the amplitude by the basic dogma of the subject for the probability of this outcome. So I need to mod square this then I've got the probability that I want. So we can work this out. We can get this complex number as soon as we know how to write plus on n as an amount of plus on z plus an amount of minus on z right?

So if we get this number and this number then we have the probability that we want is going to be mod a squared because a star is going to be exactly that number. So to get out of this ket is the bra up there that you want. By complex conjugating it you'd have a start bang in with complex z and you'd pick out a star. So the probability you want is mod a squared. So that's our exercise to find a and b and we'll be all done.

How to find a and b? Well what's the point, what's the defining characteristic of that ket? It is that it is an eigenket of this operator with eigenvalue a half. This defines n. And it's totally characteristic of these sorts of calculations of a wide range of quantum mechanical calculations that this sequence of arguments, "I want a certain complex number. It will involve some ket." Ask yourself what is the defining characteristic of the ket. It will usually be that it is an eigenket of some operator. Now we have a well defined mathematical problem.

Find it because what is sn? Sn is equal to a half of nx sigma x plus ny sigma y plus nz sigma z. A sort of a dot product between the unit vector n and the vector made up of the three Pauli matrices. Nx is zero so basically we've got an ny we agreed was going to be sine theta and this we agreed was going to be cos theta. So at the end of the day it is a half of now sigma z we've got up there. It's got one in the top left hand corner and minus one in the bottom.

So I get a cos theta and a minus cos theta appearing on the diagonal because of sigma z. And this has got a minus i in the top right hand corner so we get a minus i sine theta appearing there and its complex conjugate has to appear down here. So this is the matrix that represents sn where theta is defining the direction of n. Now all we have to do is say that this matrix cos theta minus i sine theta i sine theta cos theta on ab is equal to ab.

This eigenket this vector has to be an eigenket of this matrix with eigenvalue one in order that it's an eigenket of sn with eigenvalue of a half right? Because the original expression was sn on this equals a half of that. But here is a half I can cancel on the two sides. So I am looking for the eigenket of this operator with eigenvalue 1. Notice I don't waste my time finding out what the eigenvalues of this operator are, this matrix are, because I know because this is a matrix that represents a spin operator sn I know before I start that the eigenvalues are plus and minus, well of this one plus and minus a half of this one plus a minus one.

So we don't waste time finding out what the eigen values are. We just get on and solve these equations. There are two equations here but because we're looking at an eigenvalue problem only one of them, these two equations are lineally dependent upon one another. Only one of them contains useful information. The other one repeats that information. So we merely need to look at the top equation.

And it says that a times brackets 1 minus cos theta. So I'm going to get a cos theta equals a on the right hand side. So if I go on the right hand side we'll have a cos theta a into 1 minus cos theta is equal to minus ib sine theta. In other words we're going to have b over a which is all that I can, it's only the ratio of a to b that I can determine out of this. The absolute values have to be determined from a normalisation condition are equal to b over a is equal to 1 minus cos theta over minus i sine theta.

And we can clean this up a bit if we use some half angle formulae because this on the top is twice the sine squared of theta over 2. Sine theta is twice sine theta upon 2 cos theta upon 2. So we can cancel a number of things. The 2's cancel, one of the sine thetas cancel and we end up with sine theta over 2 over minus i cos theta over 2. So I can write now that ab is equal to cos theta over 2 i sine theta over 2. So if you work out the ratio b over a of these two I think you will get that because this minus i can be put as an i on the top.

And moreover this thing is correctly normalised. It just happens. So in principle I would now need to deal with the normalisation. I've only been calculating the ratio of the components. I want mod a squared plus mod b squared to come to 1 but it jolly well does by good fortune, right? So this is the complete bottom line. This gives you okay, right, so the probability that we pass f2 given that we passed f1 is actually equal to we said it was going to be mod a squared, is therefore cos squared theta upon 2.

Does that make sense? If theta is equal to nothing then the second filter is also measuring the z component of angular momentum. And the output from the first filter is guaranteed to return plus a half for the z component of angular momentum. So this probability must be one and indeed cos squared of nothing is 1. If the theta is pi then the second one is plus a half then n is pointing in the minus z direction. So getting plus a half in the direction n is equivalent to getting minus a half in the direction z.

But we know for certain that we're going to get plus a half in the direction z. So the probability of this happening is zero and indeed cos squared, if I put theta equal to pi I'm looking at cos squared pi upon 2 which is nothing. So that makes sense. If I put theta equal to pi upon 2 then we're measuring then the n direction becomes the y direction and we're measuring in a direction which is orthogonal to the z direction. And then you would think that knowing what components of the angular momentum of the z direction was couldn't possibly affect the angular momentum in the y direction.

So you would expect that there was equal probability the probability of passing the second filter, as I say of getting plus a half for the spin along y plus a half on y and minus on a half on y be equally likely by symmetry of the situation. And indeed cos squared of pi upon 4 is 1 upon root 2 to the cos squared of pi upon 4 is a half. And that makes perfect sense as well. So this formula predicts the kind of thing that you would expect.

Okay suppose we now have a, we won't do this in all detail but let's just sketch it out. Suppose we have now another filter. So we have f1 as before, we have f2 as we've just calculated. Now suppose on the output of f2 we include f3. So this one is going to measure in the theta direction as said. This one let's say this one has its axis in the phi direction also in the xy plane right? So you measure first of all the spin on z then you measure on the unit vector cos theta nothing sine theta cos theta, sorry. Then you measure and then those that return plus a half in that direction you measure in the direction nothing sine theta sine phi cos phi. Suppose we do that.

So the probability of passing f3 given that you passed f2 is going to be, we'll call this vector n and we'll call this vector m say, no, no we'll use this notation. This will be a half on phi a half on theta. So the output from this filter definitely has particles with plus a half component of angular momentum in the direction defined by theta. And I want to know the amplitude that those particles have will definitely give me a plus a half if I measure in the direction defined by phi. The answer to that according to the dogma of the theory is that.

And I can expand that into here. I can slide the identity operator taking the form of plus on z plus on z plus plus minus on z minus on z. We've slid identity operators in many times before in more complicated contexts. So this thing we're doing here is going to be a half phi, sorry that's a blunt end, a half phi plus z plus z a half theta plus a half phi minus z minus z and a half theta. Now these complex numbers we already know. We just calculated them right?

This was a which we used. This was b which we didn't use but we got it written down up there. It's i sine theta upon 2. So this one here is cos theta on 2. This one here is i sine theta over 2. But we also know what this is because this is going to be the same excuse me excuse me we have a complex. Let's just ask ourselves carefully exactly what is b? B is actually the complex conjugate of this, sorry. These need complex conjugate signs. Can we remind ourselves actually where we are on this? I am now worried about whether I am dealing with a complex.

Some of these need complex conjugate signs. What exactly are a and b? They were defined, okay just to get this right. What we said was that a half on theta was equal to a plus z plus b minus z. That's what we said. That was the definition of a and b. So what is this? This thing here is plus on z a half on theta. Yeah. So what I said originally was correct there are no stars here. Okay so that's just for note.

Alright now back to this. This is the complex conjugate of this is essentially the same as that with theta replaced by phi. So we know that this will be the complex conjugate of this with theta replaced by phi. This is in fact real so this is going to be cos phi over 2. Similarly this, the complex conjugative of this is the same as that with theta replaced by phi. So I now have to write down the complex conjugate of that which is minus i sine phi over 2.