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Contributor Okay so we were talking yesterday about Pauli Matrices and the way that they generalise for arbitrary spin, and I just reached this point. It's interesting to understand the connection between Pauli Matrices and the slightly strange things that happen with spin a half, and then it's also good to study the case spin one and there's a problem set problem on that which I recommend to you. And then moving right along to the case of very large spin so that we hope to recover classical mechanics, and understand how bodies which have macroscopic have many, many H bars worth of angular momentum end up pointing in some very well defined direction.

So the procedure for generating the Pauli Matrices is completely general, we're just working out the - we're just writing down a matrix, each entry of which is the value of whatever operator, here SZ squeezed between states of well defined orientation between the states. So this is the matrix made up of SM primed SZ, SM. And it's very straight forward to work out what these numbers are, they're perfectly trivial in the case of SZ because these are states of well-defined - this is an eigenstate of that operator with eigenvalue M et cetera, do we just have down the diagonal the possible allowed values of M which range from S to -S.

So this is the bottom of that matrix. And in the case of SZ, sorry SX, we replace SX with half of S+, plus S- and then we have nothing down the diagonal but we have non zero entries just on the diagonal that lies one above the main diagonal and one below, and zero everywhere else. So that's just a generalisation of the Pauli Matrix in which only this part - in the case of the Pauli Matrix only this part exists. Where this is a function alpha of this is - this S minus 1 is playing the role of M, so alpha of M, sorry this should be an M and this should be an M - is what you get - sorry this is with a raising operator so this should be M plus 1.

Sorry it's this object here. It's just some square root, so for example it's very straightforward to pick a large value of S in this diagram here which should be up there is for the case of S equals 40. And then for this large value of S to have your computer find the eigenvalues, sorry not the eigenvalues, we know the eigenvalues. Out of these three matrices I can construct if I take N is equal to for example nothing sin(theta), cos(theta), so this is the unit vector which lies in the - this is theta, this is the EZ direction, this is the EY direction so this is the unit vector N.

So I if take any unit vector whatsoever it has some coordinates like this then I get the matrix for spin down that direction being NX, SX plus NY, SY plus NZ, SZ. So choose some angle theta, take the appropriate linear combination of SZ and SY. I haven't written down SY but it's essentially the same as that with a one over 2I and some minus signs. And then have your computer calculate the state, calculate solve this problem that SN on a vector which will be a 40 component vector A1, A2 down to A40, well AS in general is equal to, let us say S, A1, through AS.

Then what are you doing? You're finding the state - the state in which you are guaranteed to get the value S, in other words the maximum possible value for the angular momentum in the direction of N and you're expressing that state as a linear combination of states with different amounts of angular momentum down the Z axis. So this number - these numbers are the relevant linear combinations. So what we're saying is that N, S in the direction of N is going to be A1 of S in the direction of Z, plus A2 of S in the direction of - sorry of S minus 1 in the direction of Z+, and so on. And actually this isn't of length S it's of length 2S plus 1.

Because there are two S plus 1 possible orientation, so if S is 40 this is going to be 81, an 81 component vector. So have it do that and what you find is what's shown in this picture up in this diagram here. This is for three different values, three different values of cos - of theta. So this is for cos(theta) is 0.5, this is for cos(theta) - sorry -0.5. This is for some other value of cos(theta) I can't quite read it, and so on. So for this, if you take this value of cos(theta) which is minus 60 degrees then these a's which of course are complex numbers have modularly that look like this.

So they're none zero in some interval around here which is to say, so what does that mean physically? What are these numbers? This is the amplitude that if you would measure along the Z axis, so first get your system into this state. Your system being in that state we would understand it to say that its spin is in the direction of theta. What then is the - this becomes the probability to measure that it has S units of angular momentum along the Z axis, this becomes the amplitude to find that you have S minus 1 units along the Z axis and so on.

So if the angular momentum - so if the angular momentum vector really were a direction theta how much would we expect to find along the Z axis? So classically in this state, theta N we expect SZ to return S cos(theta). S cos(theta) is the projection of a vector of length S pointing in the direction of theta, it's the projection of that down the X axis - down the Z axis, sorry. And what you're finding here is that these amplitudes peak around the place where classical physics would say, "This is the answer" and the quantum physics is saying, "Well you have a chance to get all these answers with probabilities which are given by the square of these numbers.

So quite strongly peaked. And as you change theta, so as you change the vector you change the state, you change the input state, you change the direction of your spin, you change the place where these amplitudes peak. So that's only for S equals 40 and classical objects have S of 10 to the 30 or whatever. And as you get more and more - as S becomes bigger there are more and more of these dots along this line here. There are here 81 dots I suppose, 81 numbers have been calculated right because there are 81 components on the vector.

By the time you've got to 10 to the 31 dots you'll find that they're really completely peaked around here. So that's how we out of this quantum mechanical stuff we recover at high spin the classical idea that things point in some definite direction. And you can go on to show that the expectation value of SY which classically should be S sin(theta) is indeed a S sin(theta). And what's more the uncertainty you can work out the RMS, you can work out what the expectation value of SY squared is and you'll find that that's essentially the same as the expectation value of SY itself squared. In other words there is very little uncertainty at high S in what you will get for SY.

So these - what's happening here is in quantum mechanics we have to calculate a whole series of numbers which are the components. So to describe the spin state of something we have to construct in the case of spin a half two numbers, in the case of spin one, three numbers and so on. Two S plus one numbers we have to calculate, being the amplitude to find the various possible answers on SZ if you would make the measurement SZ. What we're doing essentially is recovering the probability distribution for the SZ measurements which in classical physics is a delta function glitch at S cos(theta).

But in quantum mechanics our probability distributions are not delta functions, they're some kind of spread out things and you're seeing what they are there. But as you go to higher and higher spin amounts the probability distributions narrow down around the direction of spin which classically - so in classical physics we say the direction of this spin is given by the euler angles or by the polar angles, theta and phi. We just have some completely definite - oh come on you stupid thing. We have some completely definite direction and what - whereas in quantum mechanics we need a whole load of numbers because we're defining a probability distribution.

In classical physics it is strictly speaking a probability distribution but it is a delta function, and all we have to do is specify the centre point of the delta function probability distribution. And we do that with just two angles, and in quantum mechanics we need a load of different numbers to spell out the whole probability distribution properly. Now the other thing I wanted to say on this topic of relating quantum mechanical levels of angular momentum to classical levels of angular momentum is the importance of this. So we know that S squared has - the total angular momentum operator has E values S, S plus one, which is clearly greater than S squared.

And remember S, this thing came into the world as the maximum value of the angular momentum around the given axis. So - and how much greater this is than this depends on the value of S. So when S is a half we have S, S plus one is clearly equal to three quarters which is three times a quarter squared, sorry a half squared. Right this is the maximum value, and that's telling us that you can have - you always have a third of your spin down each of the three axis, if you have a spin a half particle. And the most you can ever know is whether one component is pointing this way or that way.

But we never remotely get the spin properly aligned with one axis because there will always be two units of angular momentum somewhere or other in the plain orthogonal to that chosen axis. So when we have S is one, we have SS plus one is equal to two which obviously is 2x1 squared. So now the amount of angular momentum we can have down one axis is a whole half. Here it was only a third, now it's become a half of the total angular momentum. And each orthogonal, each direction in the perpendicular plain has less than in the direction that you've chosen to align the angular momentum with.

As you go down to large values of S you have SS plus one is practically equal to S squared, because obviously S squared is going to be by definition bigger than S. And that means that we can get essentially all of our angular momentum pointing down a given axis. So the important message is from this that we're familiar with this regime where we can get something to point in a well defined direction but the atomic world works in this regime where there is always loads of angular momentum in the directions that you haven't been working on.

Okay so I now want to turn to a new topic which is the addition of angular momenta the last thing we have to do with angular momenta. So this is a very important topic for atomic physics because atoms contain - I mean the simplest atom hydrogen already contains a proton that carries a half H bar of spin. An electron has the same amount of spin and then the electron may have orbital angular momentum, it may have angular momentum by virtue of its orbit around the proton, so a generically hydrogen atom contains three units of angular momentum.

And we want to know so what are the states of the atom in which the atom has well defined angular momentum? So we're going to study and this is an application of the machinery that we introduced I guess early this term to discuss composite systems. This is a classic - this is an application of our theory of composite systems. So if you feel unsure about the theory of composite systems please go back and have another look at it because this is what we're going to be applying. So all that stuff about Einstein-Podolsky-Rosen et cetera, what underpinned that.

Because what we're going to do is we're going to consider two `[[Gyro 0:15:44]]`, we're going to have Gyro 1 has - J1 has total J - J1 - J1 plus 1 so it has M lies between -J1 and J1. So that's the rate at which this Gyro spins is fixed by some server motor or something right. And it's spinning at this rate, and we're going to have Jiro2 which obviously is going to have total angular momentum squared is this. So this will be M1 and M2 is going to lie between -J2 and J2. So we've got these two Gyros. They might be objects belonging to a navigation system and we're going to stick them inside a box and they're not going to talk to each other.

There's going to be no Hamiltonian, there's going to be no coupling - physical coupling between these two Gyros at all. But we are going to put them in a box, close the lid and then say, "So what are the states in which this box has well defined angular momentum?" and they will turn out to be - what we will find is that when the box has well defined angular momentum, if you open the lid and ask what happens if I look at the angular momentum of Gyro 1 I will get a variety of different answers. It will be uncertain what I'll find for Gyro 1 and Gyro 2 will have an angular momentum which will be correlated with Gyro 1.

So when the angular momentum of the box is well defined it has a definite amount of angular momentum and it's pointing definitely in some - well you know, the amount parallel to the Z axis is definite. When you look inside the box you'll find it is uncertain what the angular momentum of the bits are. And we'll explain physically that it's a physical necessity that that's the case, that's not mysterious. But we'll I hope make it evident that that's so. Right the moment we're going to address this kind of mathematical problem we know that the states of the box - no, no sorry.

So the states - we have two sets of complete sets of states. Complete sets of states are going to be J1M, there's a family like that, and there's a family J2, M2, sorry this should have an M1 shouldn't it. Since we know what the total angular momentum of the first Gyro is the only thing to discuss is what its orientation is. And if I consider the set of states like this J1, M1 with M1 ranging between -J1 and J1 that's a complete set of states for the first Gyro. This is a complete set of states for the second Gyro. In other words we will be able to write any state of Gyro 1 as some sum AM1 J1 M1 et cetera right.

So what are the states - what's a complete set of states of the box? It's the set of states J1, M1, J2, M2, right. We discuss that that if we have a system A and a system B, a complete set of states is obtained by taking a member of the complete set of A and multiplying it by any other member of the complete set of B. If you take linear combinations of those you get everything. In other words the box can quite generally - the state of the box can be written as some sum of B M1, M2, J1, M1. That's meant to be a pointy bracket, J2, M2.

And we want to find the states of the box, so any state of the box can be written like this, where these numbers - for suitable choices of these numbers, these amplitudes what we want to do is find the states of the box which are eigen functions of the boxes angular momentum operators. And you remember when we discuss these things, these composite systems, we had that you added the operators of systems, of subsystems and you multiplied their kets that's how it worked. So we want to consider now what the relevant operators are.

Well we're going to have for Gyro 1 we have J1 squared, we have J1Z and we have J1 plus and J1 minus the raising and lowering operators where this is equal to J1X plus or minus I, J1Y. And of course we will have the same kit of operators for the second Gyro, that's for Gyro 2 and then for the box we will have J squared which will be J1 vector plus J2 vector squared. And we will have JZ which is equal to J1Z plus J2Z and we will have J plus minus which is equal to J1 plus minus plus J2 plus or minus.

So we add the operators belonging to distinct systems here because it's a squared - this thing should be the vector operator belonging to the box squared. So we add the individual - the vector operators belonging to the individual boxes. So we have to do a bit of - these are fairly straight forward, we have to do a bit of footwork on this, so let's find out - let's expand this, we have J squared for the box is equal to J1 plus J2 dotted into J1 plus J2. And this is not - I mean there's nothing funny going on here because the operators belonging to distinct systems.

Another thing we covered in the whole composite system discussion, operators belonging to distinct systems always commute. So we can multiply this out just as if they were ordinary - weren't operators but just were ordinary boring vectors and find that this comes to J1 squared plus J2 squared plus J1.J2 twice over. This is because J1I, J2J commutator vanishes. Operators belonging to distinct systems always commute. Well this is fine, this is in our list of operators, but this is not in our list of operators, right J1.J2 is not up there so we need to write this in terms of things that are up there.

So we say J1, okay no I want to get an expression for that in terms of the things already written up here. And what I do is I say, "Let's consider J1+ x J2-. That is J1X plus IJ1Y, J2X minus IJ2Y which is going to be J1X. x J1X x J2X plus this on this will give me a J1Y, J2Y which these are two of the components of the elements that are buried inside the J1.J2. But I get other stuff unfortunately which is I get +IJ1YJ2X minus J1XJ2Y. So this I want, this I don't want. But we can get rid of this by arguing that if I write down J1- J2+, so reverse the plus and the minus, this will - everything will carry across.

The first two terms will emerge but what will happen here is that this will become - this minus sign will migrate from here to here because I've changed where the minus sign happens here. So I'll get -IJ1YJ2X+ sorry -J1XJ2Y. So when I add these the left sides, these pesky terms that I don't want will go away and I will have that J1+J2-+J1-J2+ is equal to twice J1.J2-J1ZJ2Z. Right because these two taken together make J1.J2 minus the Z bits, right, which are inside here. So now we have what we want which is an expression, I now go back to this J squared here and replace that with stuff to do with J+ and J-.

So I now write that J squared is equal to J1squared plus J2squared and then I want this so I take the +J1ZJ2Z+J1+J2-+J1-J2+. So this disgusting mess on the right expresses J squared, the total angular momentum operator of the whole box in terms of operators whose action upon the states of the box I know. That's the key thing. What I've been doing here is getting an expression where I know what every one of these operators does on those states - those states of the box J1M1, J2M2 right. I do not know what JX or JY does to those things. It makes a disgusting mess but I know what every one of these operators does to those things.

That's what I've - that's the purpose of this algebra. Okay so now a little physical argument, suppose you've got your first Gyro pointing in the Z axis, sort of aligned with the Z axis, and you've got your second Gyro appointed - aligned with the Z axis. Then you'd think that your total angular momentum would be the sum of the angular momentum of the two Gyros. Because they were both parallel to the Z axis you would argue they were parallel to each other and you would have the total angular momentum in the Z direction. So what we do now is we investigate J1 J1, J2 J2.

This physical argument suggests that this is the object J1+J2, J1+J2. So this is the state of the box in which it has this much angular momentum and all of it pointing down the Z axis on the grounds that if you take two Gyros both pointing in the Z direction surely you've got a box, surely the angular momenta just adds. We want to show that this is the case, it seems reasonable physically, is it true? We check that it is true by applying the relevant operators to both sides, right. So if I do JZ on this, I'll just say JZ on the left hand side, what do I get?

I get J1ZJ2Z plus J2Z right, because J total Z is the sum of the Z operators of the Gyros operating on the right hand side which is J1J2J1J1J2J2. So the way these composite system operators work is that this looks at this and we get J1 because this is an eigenfunction of this operator with this eigenvalue times J1J1. This stands idly by J2J2 so that's that, and then I have plus. This looks at that and produces a J2, J1J1 standing idly by J2J2 produced as the eigenket so indeed we get J1 plus J2 times what we started with J1J1J2J2. So that confirms that this object is an eigenfunction of this operator for the box with the expected eigenvalue. Yeah?

Male A couple of lines above J squared is there a factor 2 in front of J1ZJ2Z?

Contributor Yeah there probably is isn't there because this came across onto this side and we wanted a two, yes thank you very much, there is a factor, and this is about to be important isn't it? There is a factor of two there because we wanted a two J1J2 from up there and we had twice this which came onto this side of the equation. So that's that. Now we check J squared, what does J squared do when it's applied? Well it's going to be J squared on this, so J squared, I want to do J squared on the right side. And J squared we've discovered is J1 squared plus J2 squared plus 2J1ZJ2Z+J1+J2-+J1-J2+.

All that disgusting mess has to operate on - that operates on J1J1J2J2. Well this operating - this is an eigenket of this operator with eigenvalue J1J1+1 and it will then return this and we will find that this gets returned so I'll just stick it in the back as a common factor. Similarly this one looks at that and produces J2J2+J1 times itself. Then J1Z looks at this and produces a J1 times this. And J2 looks at this and produces a J2 times this. So now we have a +2J1J2 and that's the action of this operator on this product. Then J1+ looks at this, tries to raise this trailing J1 to J1+1 but it can't because we're already at the top, so it kills it.

So the J+ operating on this kills it and it doesn't much matter - it does not matter what J2- does to this because it's multiplied by nothing. Similarly when this J2+ operates on this it kills it trying to raise that J2 to one more. So the action of these two operators on this is to produce nothing and I can close the bracket just there. So J squared, actually this really should be on the right hand side. J squared on the right hand side produces this bracket times this ket which shouldn't have been written so far to the right. And we can now rearrange this because we've got two J1 J2s I can take one of those J1J2s and deal with it by putting it inside there.

So I can write this is J1J1+J2+1 so I've - to this bracket I've added a J1J2 that's one of those. And the other one I've put inside this bracket by writing it as J2 times J1 plus J2 plus 1. So this one - this J1 produces a J1J2 which is the other one of those. So this is how much I've got of J1 - J1J1J2J2. And now I can immediately see that this is JJ+1 of J1J1J1J2J2 where J is J1+J2. So that proves that the thing - it proves the conjecture that we started with, that this object is an eigenstate of the box with the eigenvalue - with a total angular momentum eigenvalue J1J2.

So this establishes - so we've proved by hard work that JJ - sorry JJ this being the state of the box is equal to J1J1J2J2 where J is J1 plus J2. That was rather hard work, the next bit's easier because we can now apply the minus operator, the J- operator to both sides of this equation. And on the left side we'll get some multiple of JJ-1 and on the right side we'll get something more interesting. So now we apply J- which is equal to J1-+J2- to both sides. J- applied to J,J produces, there's a square route here.