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Presenter(s)	James Binney
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Contributor So I think where we finished on Friday was not quite at the end of the logic of adding angular momenta – remember we had these two gyros in a box – the totals the rate at which one span was J1 the rate of which the other was spinning was J2 and we were trying to understand what the states of the box were, which had well defined angular momentum and what predictions we would get if we opened the box and measured the individual gyros.

And we had shown that what you would expect on physical grounds was the case, that if you orient the first gyro with the Z axis and the second gyro both with the Z axis also, then the two angular momenta would add because it was as if they were parallel to each other, and we would get a state with total angular momentum J1 + J2 and apparently all of it down the Z axis.

And then we used the J- operator, the reorientation operator, J- to create this state in which we still had the same large amount of angular momentum because the two gyros are parallel to each other, but we didn't have it all parallel to the Z axis.

The algebra led us to this expression here that this state is a linear combination of a state in which the first gyro is offset from the Z axis, but the second gyro was on the axis and the state in which the first gyro was on axis and second is offset from axis.

And I was just saying, as the lecture closed how to get this object here, this object here has to be a linear combination of the same two states. So this is the state of the box, this is the state of the box. Neither of these states of the boxes is a state of a well defined angular momentum of the box.

This linear combination is and there's another linear combination of these two which is a state of well defined angular momentum, that's this state which has less total angular momentum of the box and it has this state that we are looking for has to be orthogonal to that and one good way of writing it is to say that J -1, J -1 so this is the state in which the gyros are not parallel to each other, quite, but all of the angular momentum available, given that they are not parallel is along the Z axis but this is the linear combination orthogonal which you could write as J2 over J of J1, J1 -1, J2 J2 - $\sqrt{J1}$ /J of J1 J1, J2 J2 -1.

So, here we have the slightly strange thing. We have the state in which, so this is the state in which the two gyros are not quite parallel to each other, which is why the total angular momentum of the box is less than maximum.

Here they are parallel to each other, and yet when you look in here it turns out, it looks as if they're not, because one of them is aligned with the Z axis and the other isn't.

And here we have a linear combination of the same two states of the contents of the box but with a different co efficients out front, and crucially a minus sign here and that has the physical interpretation of the two gyros not being parallel to each other.

So let's try and clarify this strange situation, well get used to it I suppose as a state of affairs, by doing a concrete example. What does it look like in the very important case that we say:

J1 is 1 and J2 is a $\frac{1}{2}$. That means obviously that J the maximum angular momentum we can get is three $\frac{1}{2}$ s and we are going to have a diagram now that looks like this: so that was all in general now we're going to be looking more, we can say more concretely what we are going to have.

We're going to have three $\frac{1}{2}$ s, three $\frac{1}{2}$ s at the top here, which is going to be the same as 1, well we can just say 1 and +.

So, in using a shorthand notation here. So I've got that J1M is now going to be objects like 1, 0 and -1 because there's no need to write this down, I'm writing down the possible values for M.

M is 1, M is 0, M is -1 and J2M can be, because J2 is a $\frac{1}{2}$, I can write this as + and - where I'm writing down the values of M in the sense of + $\frac{1}{2}$ and - $\frac{1}{2}$. Right, that's a shorthand notation that makes life a bit easier.

This is just a different notation for that state, a more compact notation for that state. If we would come down here what would we have? We would have three $\frac{1}{2}$ s one $\frac{1}{2}$ right? That's because J is three $\frac{1}{2}$ s and what would it be? It would be J1 which is 1 / J which is three $\frac{1}{2}$ s (whoops I'm in danger of running out of space – let's just shave that off) it would be 1/ three $\frac{1}{2}$ s x 0+, now I want this state, which is going to be a $\frac{1}{2}$ / three $\frac{1}{2}$ s the $\sqrt{}$ $\frac{1}{2}$ /three $\frac{1}{2}$ s of 1 -.

So let's just clean that up a little bit. That's equal to $\sqrt{}$ [U+2154] of nothing + [U+2153] of 1-.

Notice the nice thing about this is, the linear combination of these states of what's in the box that we generate comes out beautifully normalised. This thing squared plus this thing squared, [U+2154] + [U+2153] comes to 1, comes out normalised automatically and that provides a nice check on your algebra. So it's good to check that this is properly normalised, because if it isn't the algebra's gone wrong somewhere.

We now have a physical prediction if you look at this state here, and what we might be talking about now that J1 = 1. That might be the orbital angular momentum electron and that J 2= $\frac{1}{2}$ might be the spin angular momentum of the electron, so we might be talking about the total angular momentum of the electron due to both of it's spin and it's orbital motion.

And if you would look inside the box, if you would examine the atom, the electron in detail when it was in this state, you would find that there was a probability of this thing squared i.e. [U+2154]rds that the orbital angular momentum and the Z erection would be nothing and the spin would be along the Z direction and there would be a probability of [U+2153]rd that the orbital angle momentum would be all parallel to the Z axis or as parallel to the Z axis as it can be, and the electron spin would be pointing downwards. Alright?

So that's the physical interpretation of this, so we have that P spin up = [U+2154]rds and P spin down, in this particular state is = [U+2153]rd. That's the physical meaning of these numbers here.

Okay. So now let's ask what's this state here? We would like to-there are some more states to find. This is the state in which the, we have 1 unit less of angular momentum than we have on the outer circle so this is the state $\frac{1}{2}$ of $\frac{1}{2}$ of the box.

It's going to be a linear combination of these two things. And it's going to be the linear combination which is orthogonal to these two things because it's an eigenket of the total angular momentum squared operator for the box, which is an eigenket of that which has eigen value different from this. So it must be orthogonal to this, by the orthogonality of the eigenkets of permission operators.

And what is it going to be? It's going to be $\surd$ [U+2153]rd of this 0+ -$\surd$ [U+2154]rd of 1 -. So in this state the odds when you look in the box of what you find are changed. In this state which has less angular momentum in total the probability of spin up is this thing squared i.e. [U+2153]rd and the probability of spin down is [U+2154]rd s. So that's the physical implication of this.

It's an interesting exercise to apply the J- operator here to generate, so if we take this state and apply the J- operator to it, on the left side we are going to get three $\frac{1}{2}$ s - $\frac{1}{2}$, which is this state here.

Let's just have a new diagram because we are running out of space there.

So here we have three $\frac{1}{2}$ s, three $\frac{1}{2}$ s - $\frac{1}{2}$ and it's going to be obtained by using the J- operators on the left and the right of that equation and I recommend that you do this. Just write down what the answer is for the moment.

It's actually obvious $\surd$[U+2153] of -1, of -1 +2, 3 $\surd$0 -.

Now it should be this because there should be symmetry between the +, this state down here physically, sorry I'll just write it in here. This state here, physically its evident this has to be, this is obviously three $\frac{1}{2}$ s and all of it anti parallel to the axis. Three $\frac{1}{2}$ s - three $\frac{1}{2}$ s. And physically it has to be that both of them are pointing down.

Right? So it has to be -1 -. Orbital momentum down, spin down.

Which obviously negative to what we put at the top there.

This thing similarly has to be physically it should be that you can get this thing by changing 1 to -1 above, - to +, 0 stays alone and + to -. And indeed it does.

So it's an exercise that I recommend that you check that when you use J- to go from here, you do indeed arrive here which is where you expect to arrive by the symmetry between + and -. And you do.

So what's happening here physically? Let's see if we can form some kind of physical picture. We can only do this to a limited extent because of the big role that quantum uncertainty plays with small spins.

But the physical idea here is that ... so let's take this three $\frac{1}{2}$s, a half state. What do we have? We have some angular momentum vector which is in some sense three $\frac{1}{2}$ s long and it's only got a $\frac{1}{2}$ of it in the Z direction. And that is some super position of the orbital angular momentum being more or less in the XY plane. And the spin carrying you up.

So you add this vector to this vector, you get this vector.

That's sort of what the first term up the route [U+2154]rd s of 0 + symbolically indicates. Spiritually indicates.

We then also have another linear combination which is $1/\surd 3$ of 1-.

Now how do we understand that? Well, we have to draw a diagram that's something like this.

So we are now combining the orbital angular momentum which is sort of vaguely along the Z axis.

Remember I stressed that when you are dealing with small spin systems you can never get the angular momentum exactly parallel to the Z axis. There is always a significant amount in the XY plane.

So this is the direction of Z, this is the XY plane. So this vector shouldn't be going straight up. I shouldn't have drawn this vector going straight up really either, that should have been at some funny angle. In fact let's improve the quality of the diagram a bit by making it not go straight up, let's make it go like that.

Then I've got - pointing down, but again it's not pointing straight down because there's always, I stressed, we spin a $\frac{1}{2}$ there is as much angular momentum parallel to each axis at all times.

This is a sort of diagrammatic representation of that expression up there. And how do you think about this is, a possible way of thinking about this physically, is to say to yourself, well...

The angular mementa of the orbital and the spin angle momenta are interacting with each other and as a result of it processing around this fixed vector. This is the total angular momentum of the box, which by conservation of angular momentum must be a fixed thing. So you can imagine that these two vectors are processing around this vector here. And here we see two snapshots of possible configurations.

So if you imagine this thing moving around like that. Now we see this, sometime later we see this, and it'll process back to that.

Now that is not really strictly speaking a legitimate proceeding, because in doing all this stuff if we never said anything what the Hamiltonian was we never said anything about, we just had these two gyros in a box and they weren't physically interacting in any way, consequently they have no means mechanically for exchanging angular momentum.

And yet, when the box is in a state of well defined angular momentum we have these results up here and we have this state of the box is a super position of these states of the contents of the box. So be aware of this picture but there is a certain amount of intuitive satisfaction in this picture and it does at least give you a physical understanding of why it is, that a state of well defined angular momentum for the box, is not a state of well defined angular momentum of the contents of the box.

Because already classically that would be the case.

And what's happened is, by insisting that the box has a well defined angular momentum, we have forced the particles to be correlated.

Because if the orbital angular momentum, or the first gyro is doing this, in order that the total angular momentum is this, the other thing has to be that. So we have forced a correlation between the two gyros or between the spin and the orbital angular momentum. And that correlation is reflected in the entanglement of these particles in the sense that we discussed, when we talked about composites systems.

In real physical circumstances like an electron, if we do have an orbital angular momentum and spin angular momentum then there is a physically coupling between the two provided by the electro magnetic field, and it is then legitimate to think about these things processing around.

The other thing that I should probably say is that this diagram doesn't really work, if you try and make this diagram work with proper lengths. You give proper lengths to this and this thing should equal this thing. And this thing should equal this thing. You won't be able to make it happen, right? And the reason you won't be able to make it happen is because this is showing something in only two dimensions and what's really happening is in three dimensions.

So you've got to imagine, so we don't know anything about what's happening in the XY plane. Those were the terms-that was the deal we did. We said we were going to have eigent functions of L^2 or J^2 and JZ. And having chosen to know something about what JZ is doing, we've given up on, we've surrendered knowledge of what JX and what JY are doing.

So what's happening in the plane perpendicular, this is the XY plane, right? It's not X and it's not Y. It's just things happening in that plane means that you can't really draw this as a two dimensional diagram. So that's why you can't make it work vectorally.

Well, I think on that we should leave the addition of angular momenta and turn to our final topic, a very important one which is Hydrogen.

So obviously atoms are terribly important. We are made of them, it's most of what we see. Here and elsewhere. And they have also played a crucial role in the development of Quantum Mechanics. Quantum Mechanics was developed in order to build models of atoms.

It is amazing that this enterprise was successful. Because even simple atoms, like an oxygen atom, is substantially less friendly dynamical system than say the solar system.

Because it contains an oxygen, contains eight electrons and the nucleus. So it's sort of the same order of the number of particles as the solar system, but it is much more horrible dynamical problem than the solar system, because the electrons attract each other much more strongly than the planets attract each other.

So the approximation, which is fundamental to understanding the solar system, that the planets move around in the gravitational potential of the sun. And we can neglect the gravitational potential of the other planets while doing that and make ourselves a model and then add in, as a perturbation, the action of Jupiter.

The forces between the planets are not negligible, they play a crucial role in structuring the solar system, but you add them in later and they're a very small approximation relative, a very small matter relative to the electric attractions of the electrons which are really jolly large.

Another problem about an oxygen atom is that the particles are moving with speeds. Speed V which is on the order of 8/137. So several percent of the speed of light. You are talking about a system which is mildly relativistic. The contribution of relativity to motion in the solar system is very much smaller when moving at 30 kilometres a second, which is less than 1000th of the speed of light. So relativistic corrections are much more important.

Another very serious problem is that these particles which are moving around in an oxygen atom are all magnetised gyroscopes. They all have spin. Significant amount of spin. Because the earth has spin, but its spin is enormously small compared to its orbital angular momentum and the earth is magnetised. But the magnetic couplings between the sun and planets, and between planets and planets is completely derisory and negligible.

And yet, it took physicists to get a pretty good understanding of the solar system was the work of the whole 18th and 19th centuries. It was with the work of `[[Bessel 0:22:11]]` the classical structure of the solar system was pretty much under control but it was Poincare, who lived at the beginning of the 20th century, pointed out there was still an enormous gap and problem about the long term life of the solar system. And the long term life of the solar system is still and active topic of discussion and it turns out to be a very interesting and finely balanced problem.

So even though an oxygen atom is very much more complicated and unfriendly dynamical system than the solar system actually it's a very much better under control.

Quantum Mechanics enables you to bring it under very much better control than even today we have brought the solar system. It's an interesting point that these systems are, in Quantum Mechanics, actually rather easier to do than the corresponding classical system. But they are none the less very complicated and we have to proceed by stages.

And what we are going to do is study, well hydrogen is of course very important. It is a nice . . . it's a tremendously important atom. But we are also going to use it as a building block for understanding atoms in general.

So we are going to talk about . . . so what are we going to do? We're going to talk about the gross structure of hydrogen like ion.

So what do I mean by this?

Gross structure. This means that we are going to have no relativity, no spin. Intimately related to relativity in fact.

We're going to have. . . we are going to be left with point spin less articles which interact electrostatics. In non relativistic mechanics.

And over here, what are we going to do? We're going to say that the nucleus, the charge on the nucleus is going to be Z x the electron charge.

So we're putting in here a number which in hydrogen will be 1, which we can make larger in order that we can discuss the motion of electrons around oxygen nuclei or other nuclei as a building block.

No spin, oh yes, electrostatics, in other words, no magnetism. And the key thing really is we're leaving out relativity. Because magnetism is relativistic correction to electrostatics and spin arises naturally when you think about electrons in the context of relativity. As I hope you'll appreciate next year.

So we're leaving out the effects which are actually quite important, but you know one has to proceed in steps.

So now what we're going to do, what we're obviously trying to do is we're trying to solve, we're trying to find the stationary states of an atom of a system which consists of 1 electron 1 nucleus with that charge.

So we want the stationary states because they provide the key to the dynamics. Usual situation. So what's the Hamiltonian under these approximations?

It's going to be the nuclear kinetic energy, the kinetic energy of the nucleus, P nucleus^2 over the mass of a nucleus, twice the mass of the nucleus + the electrons kinetic energy + the interaction energy between these two which is going to be ZE^2 over 4 epsilon 0 XE-XN, modulus.

So this is the sum of three distinct contributions to the energy kinetic, kinetic, potential.

So we want to know what does this look like in the position representation? We want to examine that equation concretely, and the way to go is to put this into the position representation. So that's to say we bra through by XE XN H... and then what we want is the position representation of this which is going to be -H `[[bar 0:27:26]]`^{2} `[[dell 0:27:28]]`^{2} with respect to XN/ 2N nucleus.

Because this is the kinetic energy operator which we know is P-H bar gradient, so P^2 is - Hbar^2 nabla^2. So what does this mean? This sub XN means this involves derivatives with respect to the components of the position of the nucleus. - Hbar^2 / 2 mass of the electron dell^2 X electron - ZE^2 /4 epsilon 0 this is already in a position representational if you like. XE – XN.

So all this stuff x upsi - oh dear I've run out of space to put it in - x upsi is going to = E x upsi. So this is the usual stuff of upsi which is a function of XE and XN. So it's a function of six variables, is XN, XE, E. It's the wave function of the stationary state of energy E.

So we've got here now, we've reduced our abstract equation to a very frightening partial differential equation in six independent variables. Because we've got the positions, the three components of XN, and the three components of XE. So we've got a PDE in six independent variables.

The astonishing thing is that we can solve this exactly and without a huge amount of sweat.

And, as in the solution of any number of problems in physics the key is to choose your coordinates correctly. That's quite generally the case, that a problem which is very frightening in general, with a clever choice of coordinates you're all done.

So we need new coordinates. We need to transform this equation to new coordinates and the ones we take are big X, which is classically the centre of maths coordinate. So that's going to be ME, XE +mass of the nucleus x the position of the nucleus / ME + MN. Alright? So that's the centre of mass coordinate.

You may say what authority have I got to use that in the context of Quantum Mechanics? And the answer is strictly speaking I make absolutely no claims to the physical interpretations of this, it's just a suggestion of something we might use to simplify the algebra. But as rational beings we know physically what that means.

So that's three new variables, because it has three components, which are linear combinations of our old variables. We are going to have another linear combination and surprise surprise; it's going to be XN - XN with separation.

So all we do now, is plotting mathematics in order to rip out of that differential equation XN and XE and insert the corresponding things with this. So let's see how this goes.

Let's do D by DXE because that nabla^2 E is sort of this operator dotted into itself. So let's see what is this? Well, the chain rule says that it's D by DX, D by DX + D by DR so this is the chain rule. Mathematics, nothing to do with physics. But because it's mathematics it's definitely true.

And this dot implies a summation right? Because this thing's got three components so this is DX1 by DXE, D by DX1 + DX2 by DXE D by DX2 etc. etc. etc.

Fortunately these partial derivatives are nice friendly things because we just have linear combinations here so I think we can easily see what this amounts to.

This is going to be ME/ ME + MN that's what this partial derivative comes to from up there of D by DX + , sorry, and what's this partial derivative is nice and simple it's just 1. So we are just going to get +D by DR.

These are shorthand for three equations because this is D by DXE1 is equal to this thing x DxDX1 +DxDR1 etc.

We want this thing dotted into itself, so what we have to do is multiply this on itself with a dot between the two, and what do we get? We get that Dell^2 X `[[sub 0:33:02]]` E is equal to; we get this thing ^{2} of course, so we have ME/ME+MN^2 D2 by DX^2.

We can write that more handily as Dell^2 with a big X, I think more clearly. And then we get this thing ^{2} +Dell ^{2} with respect to - Dell^2 we're talking about the components. Here's the usual expression for Dell^2 but using components the big X. Here's the usual expression for Dell^2 but using the components of the separation vector R and then irritatingly we get the mixed term. Because we are taking this operator and we are multiplying it on itself so we get a mixed term of this operator of multiplying the D by DR in the next bracket, and then we have this thing doing this.

So we end up with +2 of ME of ME + MN of D2 x DX DR.

So this is not very nice. Nobody wants this.

This is excellent we've got a-we've found a relationship for this which we want, in terms of this and this which are fine, but this is definitely not required and would kindly go away and we can make it go away easily by just working out what D by DXN is?

That's going to be D big X by DXN which is going to be mass N, mass of the nucleus of a mass of electron +mass of a nucleus D by D big X and then it's going to be DR by DXN which is going to give us a -1 instead of a +1 so this will be -D by DR now.

And when we square this up to work out what Dell^2 of XN is. We get this thing squared of course M.