Title	*006 Wavefunctions for Well Defined Momentum, the Uncertainty Principle and Dynamics of a Free Particle*
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Contributor So we finished yesterday with this little application of Ehrenfest's theorem which showed that on the understanding that the Hamiltonian operator is P squared over 2M plus V, the potential energy inspired by classical physics, and on the understanding that P is the operator that I claim that it is which is defined by the relation P-hat `[[on 0:00:35]]` upsi is equal to.... right so I'm claiming that the operator defined by this equation, P-hat, is a momentum operator.

It seems reasonable to take this to be the energy operator, the Hamiltonian. That being so, when we use Ehrenfest's theorem to find the rate of change of the expectation value of X, which in classical physics would be the actual value of X, we find that it's in fact equal to the expectation value of the momentum divided by M which is in a classical sense what we would call the velocity. So that's one good thing.

It's obvious now that we should move forward and calculate the rate of change, using Ehrenfest's theorem to calculate the rate of change of the momentum's expectation value and live in the hope that this becomes the force. Anyway this is going to be – no maybe we'll leave the IH bar, put it over here same as the...

So this is going to be P comma H, expectation value of Ehrenfest's theorem over IH bar because I haven't put the IH bar here now on second thoughts.

So we need to calculate P comma H. P comma H is P comma P squared over 2M plus V, comma there. Obviously P commutes with itself, so to forget that and so therefore this is P comma V and when we discussed commutators we showed that if you take the commutator of an operator with a function of an operator – this is a function of X – then what you end up with is the derivative of this operator.

Well you do end up with that in the event that the commutator... so do you remember we expanded – what we did was we expanded V of X as V0 plus V1X-hat plus V2X-hat squared over 2 factorial etc. etc. etc.

And then when we calculated P-hatV what did we get? We got V0 plus V1P-hat etc. plus – and here we would have V2 over 2, this would be because we're taking the commutator of P with X squared, which is P with X, with the other X standing idly by, plus XP comma the other X, from our basic rule for doing the commutator with products.

Because this thing is only a number, it's minus IH bar in fact, we can take this number outside, it doesn't matter the fact that this number's in front of X and this number is behind, here it's behind X, because it's a number we can just pull it out.

This becomes 2X-hat which cancels this and at the end of the day we are looking at P comma, P comma X, a common factor in all these things plus – sorry, brackets, V1 plus V2X plus V3X squared over 3 etc. etc. etc. Sorry, over 2, which is the Taylor series for DV by DX and this one here is minus IH bar so we have minus IH bar DV by DX.

So our equation of motion – so putting this commutator back in up there we discover that D by DT, the rate of change of the expectation value of the momentum is, oops we pick up a minus sign from here because we have a minus IH bar here and we want to find the commutator over IH bar.

So we get minus the expectation value of DV by DX. So lo and behold we have Newton's law of motion. We have the rate of change of momentum is equal to force, but in this expectation value sense.

It's the expectation value of the rate of change, the rate of change of the expectation value of the momentum is equal to the expectation value of the force because in some sense the force has to be thought of as something that's – well it is, it's something that has quantum uncertainty because it has uncertainty because the position is uncertain. Different positions will give rise to different forces etc.

So I think that makes a pretty convincing case that the momentum operator is as advertised because we're able to recover all that understanding Newton's laws of motion.

So now let's look at states, a very important topic. Let me do it here in fact. States of well defined momentum, that is to say we want to know what are the wave functions, what do the states look like in which you're certain to – the measurement of the momentum is certain to produce a given number?

Okay so we're interested in the eigenstates of the momentum operator. The operator P on an eigenstate of P, on an eigenstate labelled by P – this is a number – is equal to that number times P. So this is the definition, this defines these states.

If we want to know what these things look like in real space we want to bra through with an X and then we're looking at XP-hatP is equal to PXP, this is the wave function of our state of well defined momentum.

Let's introduce a newfangled notation and declare that this is U sub P of X. This is just the definition... the wave function. XP equals UP of X. And this left side by the definition of the P operator is minus IH bar DUP by X.

So here we have one of these trivial differential equations which we know how to solve. It tells us that UP of X is equal to a constant times E to the IP over H bar X. If we put E to the IP over H bar X in for U, when we do this differentiation we get down an IP over H bar, the H bar's cancel, the minus I and the I together make a 1 and the P sticks around, is what we want, so that's it.

So a state of well defined momentum, the states in which you are certain to measure a given value of the momentum is a plain wave, is a wave like this. So it's a wave and we have the wave number, usually called K, is the momentum divided by H bar.

Because H bar is incredibly small typically this wave number is extremely large and the wavelength of course lambda being 2Pi over K is 2PiH bar over P is H over P is going to be very small. And the bigger the momentum the smaller the wavelength, that's obviously crucial for physical applications.

What else can we say? We can say that there's complete – if you know the momentum then – so if we're in a state of well defined momentum, the result of measuring momentum is certain, so you do know the momentum, then your wave function looks like this, which means that the probability density is independent of space.

So the probability density which is UP squared is equal to some constant which is independent of X. In other words you know absolutely nothing about the location of your particle, absolutely nothing. It's as likely to be here as on the other side of the universe.

So from that it follows you've already got – it's like these states of well defined energy, these states of well defined momentum, do not in practice occur, they are mathematical idealisations. Because you would never see a particle which had totally uncertain position because it would spend all its time not in your laboratory, your laboratory's such a negligible part of the universe. Okay, so that's something to be a bit clear about.

What else did I want to say about this? Oh yes, we should address this wavelength, I should mention this of course, is called the De Broglie wavelength. De Broglie was thinking about relativity in 1924 or whatever in his thesis for which he won the Nobel Prize in 1929 and he came up with the idea that there was this relationship between – that the particles would be associated with a wavelength. So that's called the De Broglie wavelength in his honour.

And as regards numbers, well we'll look at some numbers later on but the general idea is that the size of an atom is determined by the De Broglie wavelength of the electrons that make up atoms.

So if you have a hydrogen atom in its ground state its characteristic size is given by the De Broglie wavelength of the electron that's in there and the electron that's in there is in orbit around the proton with a certain momentum.

Right so this De Broglie wavelength is setting the size of atoms, I think that's a point worth making. But we'll look at some numbers later on.

So if you have an electron, so an electron in hydrogen right is moving around, it has – the binding energy of hydrogen is 13.6 eV, it has a kinetic energy which is half that because of the virial theorem which – we'll have all these results later on but they're already in classical physics – so it has a kinetic energy of the order of 6 eV and that gives you a De Broglie wavelength which is a tenth of a nanometer. That gives you some kind of sense of scale.

Okay what about normalisation? So we've deduced that the wave function of a state of well defined momentum should be some constant times this exponential. It's good to decide what this constant should be.

We usually normalise our wave functions – so usually we want to have, we like to have that the integral DX of upsi mod squared is 1, because that's the total probability to find it somewhere. But this normalisation isn't going to work because if upsi is proportional to E to the IKX, upsi mod squared is going to be 1. The integral from minus infinity to infinity of 1 is just infinite and no constant in front is going to normalise it successfully.

So we don't use that normalisation, the normalisation that we use is this normalisation. But do you remember yesterday we agreed that X primed X should be delta of X minus X primed.? So this thing here is the amplitude to be at X primed if you're certainly at X which is why it's nothing unless X primed is equal to X. And this amplitude becomes very large when X equals X primed so that when you integrate over this you get 1.

So that's what we should do in this case. P is an operator with a continuous spectrum, same as X. So we want to choose the normalisations constant, choose the constant, such that P primed P equals 1, sorry not 1, delta P minus P primed, by precise analogy with that.

So that's something that's fairly straightforward to do, we write this – we put an identity operator into here made up of Xs. So this implies that – well this thing here is equal to P primed XXP, that's just sticking in an identity operator. We're going to say that XP is equal to some normalising constant times E to the IP on H bar X.

And the name of the game is to find the value of this because we know that this thing is this. The nice thing is that this is the complex conjugate of that so what we have is that this is equal to A mod squared because we get an A from here and an A star from here. The integral DX of E to the minus IP primed over H bar X, that's from here. The complex conjugate of that with P made into P primed and from this we simply have an E to the IP over H bar X.

And that can be written, just to clean it up a little bit, P minus P primed X over H bar DX over H bar, H bar. So this H bar was always present, this one I've put in. I've divided the X by H bar and multiplied by compensating H bar here so the variable of integration is now X over H bar which is still running from minus infinity to plus infinity and now this is a standard integral which I hope you all recognise from Professor Essler's course, from Fourier analysis. From Fourier analysis we know that this integral is 2Pi times delta P minus P primed.

So what we're concluding is, going right back up to the top, that that original delta P minus P primed right up there is equal through these integrals to mod A squared times 2Pi H bar, 2Pi H bar delta P minus P primed and that clearly tells us that A mod squared is equal to 2Pi H bar, is just H is equal to 1 over H and the phase of A is unimportant so we're entitled to take it to be real. So what we do is we choose A, do 1 over the square root of H, not H bar but H.

So that means that the correctly normalised thing X wave function XP is E to the I, P over H bar X over the square root of H. So this is an important result.

It tells us something else that's of interest if we take its complex conjugate, because its complex conjugate says that PX is equal to E to the minus IP over H bar X over root H. What does this mean? This means the amplitude to find that you have momentum P given that you're definitely the place X.

So if you have an electron that's localised to the place X its wave function is a delta function essentially right, it's localised at. You can ask what's the amplitude for this to have various momenta? The answer is given by this complex number here.

The modular, so this complex number here, is independent of P. So what does that mean? It implies that the probability of having P given X is some constant. All values and momentum are equally likely, from a momentum which is nothing very much, or zero even, up to a momentum which is associated with some relativistic gamma, some large value of gamma. All momenta are equally likely including extremely high ones.

So that's clearly unphysical and what that tells you is you will never succeed in localising a particle precisely to an exact X. The state of being definitely at X is unrealisable because it would imply that there was enough energy somehow in the system that there was a non-negligible probability of finding the momentum to have some extraordinarily large values.

Right so there we are, that's the – so what we've discovered so far is if X is certain P is totally uncertain and conversely if P is certain X is totally uncertain.

Let's therefore investigate – both of these situations are clearly unphysical, so let's try and discuss something which is physical and let's suppose that we're dealing with a probability distribution in X which is Gaussian, E to the minus X squared – oops – X squared over 2 sigma squared over the square root 2Pi sigma squared.

So this is a Gaussian distribution of probability in X which is our generic model of well we've got this thing localised at the origin to within plus or minus sigma more or less.

We can ask what wave function yields this probability, well the answer is essentially it's a wave function which is the square root of this so a suitable wave function. There are many possible wave functions because phase information isn't conveyed by the probability but let's write down this wave function which is E to the minus X squared over 4 sigma squared over 2Pi sigma squared to the quarter power.

So if you take the mod square of this wave function you get the probability and the probability you get to that one there. So I could multiply this by all kinds of complex – all kinds of numbers of modulus 1 and arbitrary phase and I would still get that. But this real wave function is the simplest one that we can write down.

And now let's calculate for this. So this is a well defined wave function which we know localises our particle to the origin plus or minus sigma. Let's ask so what is the probability distribution for this upsi of measuring a particular value of P? So what we want to discover for this is what's Pupsi?

Well that's the integral DX of PXX upsi. We know what this is because we've just been working it out. This is a state of well defined momentum. So this is the integral DX of E to the minus IP upon H bar X I believe – I hope I've got that minus sign right somewhere up there – over the square root of H times this which is the wave function we just wrote down, E to the minus X squared over 4 sigma squared over 2Pi sigma squared to the quarter power. And we have to integrate this from minus infinity to infinity.

Now physics is full of integrals of this sort and there's a box in the book explaining how to do them. I don't want to take the time to go into the sordid details now. But all you do is you gather all these exponents of the exponential together and what we've got here is an integral DX of E to the I quadratic in X.

If you gather this together there's a linear term and there's a quadratic term so you can express that – I mean it is E to the I quadratic expression in X – and what you do is complete the square of the quadratic, change your variable of integration and use a standard result that the integral DX E to the minus X squared from minus infinity to infinity is equal to the square root of Pi. We use this standard result. And that's how we evaluate these integrals here.

But I would recommend checking the box out, making sure you understand how that goes and doing this integral yourself as an example after the lecture. But I don't want to take time to do it now because it's just algebra. Let's just write down the answer and discuss its physical implications.

So this turns out to be that Pupsi is equal to E to the minus sigma squared P squared over H bar squared over – and there's a normalising constant which is 2Pi H bar squared 4 over 4 sigma squared to the quarter power.

So if we square this up we get the probability of measuring various momenta which is clearly going to be E to the minus 2 sigma squared P squared over H bar squared over 2PiH bar squared over 4 sigma squared to the quarter.

So our probability in position in real space we have the particle localised in a Gaussian distribution with a width sigma. It turns out from this calculation that the possible values of the momentum, the probabilities associated with different momenta is also a Gaussian distribution centred on zero in momentum.

And the width of this distribution, the spread in momentum – so in order to find what that is you'd have to express this as E to the minus P squared over 2 sigma P squared. So the dispersion in momentum is H bar over 2 sigma. So the dispersion in momentum is small when the uncertainty in real position is large and conversely.

So we have a result that for this particular model the dispersion in X times the dispersion in momentum is H bar over 2.

Male 1 Professor?

Contributor Yes?

Male 1 `[[?? 0:26:51]]` probability to measure `[[?? 0:26:53]]` a half `[[??0:26:55]]`?

Contributor You are worried about this 2?

Male 1 Below there you've got `[[?? 0:27:04]]`.

Contributor Oops thank you it should be a half, yes of course, because I've squared the quarter and it's become a half. So this is the classical statement of the uncertainty principle.

It's really only in order of – in this particular model, this is an exact mathematical statement. It's a statement about the widths of two Gaussians. But in a generic case, if you know your probability distribution is like this, just some curve that's sort of got a natural width and a location in X then the corresponding probability distribution in P will have a width which is broadly related to the width in X here by a relationship of this type. But it won't be exactly H bar over 2 in the generic case. It's exactly H bar over 2 just for these Gaussian distributions.

But the really key idea is that the product of the uncertainties in these two things will be on the order of H bar.

So there are two important points to make here. We need to be clear what we're saying. We are not saying that if you measure the position of an electron and then you measure its momentum you will find results which scatter in this way. This is not the uncertainty of a measurement in X and then the uncertainty of the following measurement, the following momentum measurement.

This is a statement about if I have a large supply of different electrons set up so that they're pretty much in the same wave function and I choose to measure the momenta of half of them I'll get a dispersion sigma P and if I measure the positions of the other half of them I'll get an uncertainty sigma X which satisfies this relationship.

Because we have – this uncertainty in momentum is the uncertainty associated with the original wave function Xupsi and if I would measure the position of that electron I would change the wave function into some kind of a – something near to a delta function, centred on whatever answer I got.

So when you make a measurement you change the wave function and we've calculated the dispersions for measurements using the same wave function not an initial wave function and then the wave function that we get when we make the measurement.

And the reason we've done this partly is that we do not know what the wave function is we get when we make the measurement, that's in the lap of the Gods. You make a measurement – so remember the basic dogma. Let's go back to the discreet case because it's simpler. If I have my wave function is some sum ANEN, some linear combination of stationary states, this is a well defined wave function.

If I measure the energy then this thing collapses to upsi is equal to EK for some K and which K is in the lap of the Gods. The apparatus does not tell us it just – you know the roulette wheel is spun and one of the Ks is chosen.

So it is up here, if you measure the position you will find some value and after you've made that measurement your wave function will be different, it will be more or less a delta function associated with that X, not the wave function we're working with here. And the uncertainty on a subsequent measurement of P will be larger... will be large.

The other thing to say is how do we understand this physically, this uncertainty relationship? We say to ourselves well if the wave function is highly localised in space – if you think about that wave function as made up as an interference pattern between states, between plain waves, which are states of well defined momentum – then in order to have the interference pattern highly localised so that the sum of all these waves cancels to high precision everywhere except in some narrow region, you will need to use waves with a very large range in wave numbers. And that's why the momentum is very uncertain if the position is rather certain.

So because of this basic principle of adding amplitudes - a highly localised electron, we're entitled to think about a highly localised electron as an interference pattern between states of different momenta and we will need to have a very large range of possible momenta if we want to have a highly localised electron and tightly confined interference `[[?? 0:32:40]]`.

Let us now talk about the dynamics of a free particle. So we've just got a particle whose energy – there's no potential energy, it's just free to roam, so the Hamiltonian operator is going to be P squared over 2M, we drop the plus V of X, it's a free particle.

And what we're going to do now is talk about the time evolution of this particle. So imagine that you've got the particle and that T equals 0, you've got it localised around the origin and let's zazz this up a little bit by saying it's localised around the origin but it's moving with some, you know, we've got some idea what its velocity is.

So we're going to say it's initially – we're going to write down an appropriate expression for its momentum. So this is the wave function in – well it's a complete set of amplitudes with respect to momentum of a particle which is localised at the origin and has no – the mean momentum is nothing.

Suppose we start from Pupsi is E to the minus sigma squared over H bar squared, P squared minus P0 squared, sorry, P minus P0 – sorry, what do I want to do? Yes, P minus P0 squared over this horrible normalising constant, 2Pi H bar squared over 4 sigma squared a quarter.

So it would be reasonable to conjecture that – we'll find out whether this is true or not when we do the calculation – but the conjecture is, the reasonable conjecture is, that this complete set of amplitudes characterises a state of the particle where it is moving with momentum P0. P0 is a constant right? This is the momentum eigenvalue, this is just some constant.

So it has a velocity which is on the order of P0 over M and it's localised at the origin to plus or minus sigma. We'll find out whether this is true or not but that's the conjecture.

Now let's ask ourselves what is the wave function in real space that corresponds to that at different times, as a function of time, why can we do this? Because we have a free particle, the Hamiltonian is just P squared over M which means that a state of well defined energy is going to be a state of well defined momentum.

The Hamiltonian is a function of the momentum so it has the same eigenstates as the momentum, so a state of well defined momentum is going to be an eigenstate also of the energy.

Now we know how to evolve in time states once we – so remember our basic equation which is that upsi at time T is equal to the sum ANE to the minus IENT over H bar times EN at 0. Remember this was why we were excited by the states, why these states of well defined energy, the stationary states are so important is because they enable us to evolve in time a system where AN is equal to EN0upsi. These things set the initial condition for the calculation and the time evolution is given by these exponentials.

So we want to use this formula in this other context here. We know what this is; this is a state of well defined momentum. We know what this is, this is just some exponential with the relevant energy going in there. And this is the amplitude to have momentum P.

So this transforms – this is the discreet case – this transforms in our case into upsi is equal to an integral over all possible momenta – that's the analogue of the summing over the energies. When you sum over momentum you are summing over energy, because different momentum... alright?

E to the minus I, what's this? This is the energy associated with momentum P, I called it EP up there but we can be more definite, it's P squared over 2MH bar T, sorry T over H bar, excuse me, T over H bar. That's the exponential thingy there and what's this got to be? This has got to be a state of well defined P.

We wanted to know what this looked like in real space so let's bra through with X and then this... sorry, I'm missing something altogether, excuse me. Let's leave that out, I've missed something out.

I missed out the ANs didn't I? What are the ANs? It's the amplitude to have – at time T equals 0 – is the amplitude to have energy EN, which in our case is the amplitude at T equals 0 to have momentum P. And then now we have the state P. And now if we want the wave function information we should bra through with X.

Then everything over here becomes a function of momentum and a known function of momentum. This is a function of momentum, also time. This is a function of momentum we just put it down by conjecture, it's that thing there. This is a function of momentum, this is a plain wave, this is E to the IP on H bar X within a `[[sine 0:39:21]]`. No it is exactly that.

So let's just see what we get here. So this is a dirty great integral, DP, E to the minus I, P squared T over 2MH bar. Let's put this one – no, keep to the right order – E, then here we have E, what we said it was going to be, E to the minus sigma squared over H bar squared P minus P0 squared over a horrible 2Pi H bar squared over 4 sigma squared to the one quarter power, if I've got that right.

And this thing is our wave function for a state of well defined momentum which is E to the I, P over H bar, sorry PX over H bar, over the square root of just H.

So what do we have here? We have an integral of an exponential of a quadratic expression in P because here we have a P squared. When you square this thing up you're going to have a P squared and a minus 2PP – and a linear part in P. And here's a linear part in P. So it's another of these integrals of an exponential of a quadratic expression in P which can be solved by the methods described in the box that we used just before.

Now the algebra in this case is a little bit wearisome. It's absolutely straightforward but it's just a bit wearisome. And the answer in fact that this comes to is quite a complicated expression because what we're going to arrive at is something which has both phase information and amplitude information.

But we only want to know what the probability is of finding the particle at this place or the other place. And that probability, the mod square of the answer to this calculation is much simpler and I'm going to write it down.

So what follows now is a very straightforward calculation. I would urge you – there's a box doing it in the book – I would urge you afterwards to look through this and make sure you understand it. But it is just algebra and what's interesting is to understand the physical implication of this.

So we're going to extract the mod square of the answer when you've done all this integration. And what apparently it is is sigma over root 2Pi H bar squared mod B squared E to the minus X minus P0 T over M squared. And I need to tell you what B squared is don't I so in here B squared is a complex animal, it's sigma squared over H bar squared plus IT over 2M H bar.

So what have we got? This is a Gaussian distribution in X, at any fixed time it's a Gaussian distribution in X. The centre of the Gaussian is that P0 over M times time, which means that it's centred on what one would call V times time right?

Because P0 over M we said this was the mean momentum of – it was the expectation value of the momentum of our original wave function. So it's the mean – if you thought of this as many different particles, it's really only one particle, but if I thought of it as many different particles it would be the mean momentum. So this is essentially the mean velocity.

So that's what you would expect. The probability distribution is moving in space with the speed V0, equal to P0 over M as we would expect and the dispersion associated with this Gaussian is determined by that stuff.

So we have a sigma as a function of time which is going to be given by... so what should this be? This should be 2 sigma squared. So sigma is going to be given by the square root of those two which is going to be from this sigma squared plus T squared – I'd better write this down, it's too hard to do it in one's head. Plus H bar T over 2M sigma.

Male 2 Is this the same sigma as in the `[[?? 0:44:27]]`?

Contributor Yes sorry so this should be another sigma, what should we call this? Well let's just call this the dispersion. Or we can call it sigma sub T, right, whereas this other sigma is the original sigma. So we've got a Gaussian distribution and that has a dispersion given by this. No, sorry there's something wrong here isn't there because on dimensional grounds...

Male 3 `[[?? 0:45:10]]` sigma `[[??0:45:11]]?`

Contributor Did I write down the right integral? No I didn't. That's exactly what's gone wrong. Sorry, we're missing from here a sigma squared on top, that's crucial.

So when I say what the dispersion should be, we should arrange this is 2Pi dispersion squared, so dispersion squared is equal to this divided by that. Sorry, then I have to square root it so it's divided by that which makes it that. And this I've copied out of my notes so I expect it's still right but I was trying to do some of this in my head which was dangerous.

So what have we got? We've got that the dispersion at time T is equal to the original dispersion at time T0 plus this extra bit here. And what is this extra bit here, what was the original uncertainty? The original uncertainty in momentum from the uncertainty principle here – the uncertainty momentum was equal to H bar over 2 sigma.

So the uncertainty in the velocity was equal to H bar over 2M sigma, so what's this? This is equal to sigma plus the uncertainty in the velocity times time... squared. I shouldn't be squaring this, should I? I think we might need to take a square root of a square actually. Let's not chase that down at the moment, because this is the basic idea.

The basic idea is the uncertainty in position is growing like the uncertainty in position times velocity but that's what you would expect, right? Because you have – what do we have? We have a bunch of particles originally at the origin and moving to the right with V0 plus or minus delta V. Some are going faster, some are going slower.

At some later time this is moved over by an amount V0 times time and this width of – there was a width sigma here but the ones that were going slower than the average will have slipped behind. Some of them will already be sigma behind but then they've slipped behind extra by an amount delta V times T and some of the ones which were in front have got even more in front because they had bigger velocities by delta V.

So the total width is equal to the original width plus this extra width and I think probably we should be taking some squares and square rooting.

But you see that what we are getting from this calculation makes perfectly good sense physically. And let me just remind you how we've done this calculation because it's the methodology which is in many ways – well it's good to see, it's crucial to see that what emerges from this makes sense physically but it's also good to remind yourself how do you actually calculate these things and this damn theory.

The way we've done this is we've used this central expression. We've said that states – I can evolve something in time so long as I can express my original state as a linear combination of states of well defined energy. In this particular case of a free particle a state of well defined energy is exactly the same as a state of well defined momentum so we wrote that sum expression in the integral form that's appropriate because momentum has a continuous spectrum. And then we just turned the handle and out came these perfectly sensible results.

I think we're probably pretty much ready to finish. Again I want to stress – I think I should stress – that we've obtained this perfectly sensible physical picture through an orgy of quantum interference because we have – in order to get what we wanted we took a perfectly well defined spatial distribution and expressed it as an interference pattern between states of well defined momentum.

Which we then evolved each state of well defined momentum in time, in its trivial way, that's just an exponential. And then we allowed them to interfere at this later time in their evolved form to find out what the distribution was in real space.

So that's what I mean by it's an orgy of quantum interference. We've taken something; we've decomposed it into an infinite number of other things. We've taken something physical; we've decomposed it into an infinite number of things which are not really very physical, mainly states of well defined momentum.

We've evolved each one of those independently in time because they're states of well defined energy and then we've interfered the evolved momentum states, we've allowed – by working out this integral was working out the result of the corresponding interference.

We were adding up an infinite number of amplitudes and allowing them to interfere and out comes something that makes sense which is a wave packet that's travelling and spreading and behaves in a way which does make perfect sense from a physical point of view, from a classical physical point of view. Okay we'll finish with that.
