Title	*008 The Harmonic Oscillator and the Wavefunctions of its Stationary States*
Description	Eighth lecture of the Quantum Mechanics course given in Michaelmas Term 2009
Presenter(s)	James Binney
Recording	http://media.podcasts.ox.ac.uk/physics/quantum_mechanics/audio/quantum-mechanics8-medium-audio.mp3
Keywords	physics, quantum mechanics, mathematics, F342, 1
Part of series	*Quantum Mechanics*

Contributor Okay, so this, we were at work on the harmonic oscillator which as I said, is the single most important dynamical system in physics; a model for a huge number of physical applications. We had this Hamiltonian which is basically P2 plus X2. The crucial think is that it's quadratic in, which obviously quadratic in P. But specially it's also quadratic X, that's its peculiar characteristic of its potential energy. We went, the plan for, what we're trying to do is find the stationary states. In other words, we're trying to find the states here, which are Eigen functions of the Hamiltonian, because they will enable us to follow the dynamics of the system.

They're crucial tools. We went after them by defining this thing A, the annihilation operator which is a dimensionless operator made up of X and P. And we found that if the cog, we evaluated, well we found this thing had two properties. First A dagger A is very nearly the Hamiltonian A dagger A plus a half H bar omega is the Hamiltonian, the first important property. The second important property that its commutator with its commission adjoint is minus one when done this way round.

So using those properties, we said suppose we have a state, a stationary state of energy E. If we multiply this equation through, this defining equation of that state through by A dagger and do some swapping of the order of operators using the results above, we were able to show, this is where we finished, that H on the state that you get by using A dagger on E is equal to E plus a H bar omega plus H bar omega times that self same state. In other words, this state here, is essentially a state, well it is a state, a stationary state of energy, increased by H bar omega.

And because we could repeat this, it follows that the energies, if there is an energy E, then there must be energies E plus H bar omega, E plus 2 H bar omega. And as far as we can see, E plus any amount, any number of H bar omegas. And what remains at this point, is to find out what the number E is, that we first thought of. And we do that by applying not A dagger to that equation, but A to that equation. So then we have, if we take, if we start with EE is equal to HE and we multiply both sides of the equation by A, not A dagger this time. Then we swap the order here, just as we did before into HA.

And then we have to add in the commutator A comma H, all operating on E still. This is absolutely a repeat of what we did yesterday. This is HA. Now we replace that H as advertised up there by A dagger A etcetera. We observe, so this should become the, this is going to be replaced by A dagger A plus a half H bar omega. We can forget about the half because we're inside the commutator and a half commutes with everything. We can take the H bar omega outside the commutator and then what we're looking at is plus A, A dagger A, H bar omega, close brackets E.

We use our usual rules for taking the commutator of product, which is to say it should be the commutator of this with this, with that standing idly by. And then in principle the commutator of this with this with this standing idly by, with the second commutator vanishes. And this commutator, by the result up there, is equal to plus one okay, because we turned around there A dagger A as minus one so this commutator A,A dagger must be plus 1. So this is saying this becomes HA plus A, H bar omega E. So we have an AE with no operator in front on this side of the equation. But we have the same thing on that side of the equation.

So we take these together and onto this side of the equation and we then have that E minus this H bar omega goes to the other side and becomes a minus H bar omega A,E is equal to HAE. Which establishes on the face of it, what we might have hoped, which is that the state you get that you have after you use A on this stationary state E, is another stationary state with an energy lower than E by H bar omega. So A dagger raises your energy, it increases the excitation of the harmonic oscillator and A evidently lowers the energy of the harmonic oscillator.

Now you have to at this point start to worry because by the previous rhetoric, it would follow that if there's E, then there is also E minus H bar omega and so on, down through E minus N h bar omega. And it looks as if you can find states of lower and lower energy without limit. And you should be worried about, because thinking about the Hamiltonian up there, this classically it looks manifestly positive. Because P2 should be positive and X2 should be positive. You can't trust such things in quantum mechanics.

But what we can do, is we can work out, we can say look, E is the expectation value of the E Hamiltonian in the state E obviously. Because H on E is equal to E, it comes outside and this on this is one, so that's a self evident equation. Replace what's inside here, this becomes over two M of E P P E plus M2 omega2 E X X E and because these are observables and therefore omission, I can put a dagger there if I want to. It doesn't make any difference because P dagger is P and I can observe that this is now manifestly P E Mod2.

This thing is the Mod2 of that, of the ket you get by using P on E and this is M2 omega2 of X E Mod2 still over two M which is boring. And this is manifestly greater than or equal to zero. Even in quantum mechanics, there can't be any argy bargy, this has to be greater than or equal to zero. So all the energies, all the allowed energies, the entire spectrum of the Hamiltonian has to be positive. But we've apparently established that by using A, we can get states which, by successively using A and A and A and A, we can get states of lower and lower energy. Because we take H bar omega off every time we use E, sorry we use A.

So there's a problem. And the problem is, the assumption. So let's just look carefully at what we've established here. What we've established, is this equation here. This equation is absolutely copper bottomed. It's beyond criticism. It's true. It was obtained by totally legitimate operations. And it establishes that this thing is an Eigen function with this lower energy, provided this thing is non zero. But if at any stage in this chain of applying A's, this thing here would vanish. So if, when you get to a certain energy, E lowest energy and you apply A, A simply kills it. It produces a ket of no length at all. Then we won't have a state of even lower energy.

And since it's clear that this chain of operations, it's logically impossible to go creating states of lower and lower energy. This chain of results of applying extra factors of A has to stop somewhere. And the only way it can stop, since this equation is true, is by this thing becoming zero. Then, because H on zero is equal to any number you like on zero. So if this is zero, this equation does not establish that this is an Eigen value of H, right. But that's the only circumstance in which this equation would not establish that this was an Eigen value of H.

So for the lowest energy, the ground state energy, it has to be that A kills it. So that's the ground state. We must have that A on E zero equals nought right. So I'm using this symbol now to indicate the lowest energy. So what does this equation mean? To give more precise meaning to it, so what we say is that this ket that you get here, has no length squared. So let's just evaluate the Mod length. So we're saying that A is zero Mod2, which is equal to E A dagger A E is nothing.

But this thing, we had it somewhere up there, it's just, yeah it's just in range. H is equal to A dagger A plus a half H bar.

So this thing is, this is the expectation value of H over H bar omega minus a half. Sorry, these want to have zeros on it, sorry these wants to have zeros on it because we're talking about the ground state energy, not any old energy now. We're just talking about that one special lowest energy, the ground state energy that this equation is valid for. And so what we're discovering is that E zero over H bar omega is equal to a half. In other words, the ground state energy E zero is a half H bar omega. And now we know what the general energy is, because we know that we can make states of higher energy by applying A dagger to the ground state

Moving up by our H bar omega each time, the nth energy must be N plus a half H bar omega. So we have found the allowed energies and the next item on the agenda is to find the corresponding, find the wave functions of the corresponding Eigen states, stationary states. I'll just move over here. So let's look at, let's ask about wave functions of the stationary states now that we've found the allowed energies. Actually before I do that, it's probably good to generalise this calculation here. So what we've established is, so let's just talk about normalisation as a sort of preliminary. I think it's better to do this now. Normalisation.

What we've established is that A dagger on, and I'm going to introduce a new notation, right. We're going to say what previously I called E, is going to go to N. So this is the state with E equal N plus a half H bar omega. Right, we could write EN in here, but everybody writes just N, it's just saves energy, it works well. Right, now we've discovered what the energies are and that they're labelled by an integer N nought, one, two, three, four okay. So this is N equals nought, one, two and so on. It makes sense to have a nice compact notation and to call our states, the stationary states, the state nought, the ground state, state one, the first excited state, the state two the second excited state.

And what we've established is that A on N is some constant, I'll call it K times N plus one, right. Because we've discovered that when we used A on the state E, we got a state which was an Eigen ket of H with Eigen value E plus H bar omega which would be in the new notation the state N plus one. If this E was N plus a half H bar omega, this will be N plus three halves H bar omega. And the issue arises, so what value does this thing have? So just to be clear, so when we have, we've had relationships like this, we had a corresponding one up there, yep, there we go.

Yesterday, we derived this equation and that equation establishes that this thing is an Eigen ket of H, it doesn't establish that it's a properly normalised Eigen ket of H, I want it to be properly normalised. And so I'm asking about what's the normalisation constant that I have to use, after I have applied an A dagger. It's easily found because we just take the Mod2 of this side of the equation and the Mod2 of that side of the equation. So taking the Mod2 both sides, we get N A A dagger N is equal to K Mod2 times one because that's by definition, going to be correctly normalised.

We can take K to be a real number, we can take K to be a complex number if we determine to, but why don't we just agree to take it to be a real number and this just becomes K2. We're just trying to get the thing normalised, we don't care about the phase. And let's ask ourselves what this is. If I swap this over, I want to swap this over because then I can relate it to the Hamiltonian. Then I have to add on A,A dagger commutator. So that's that rewritten. This is the Hamiltonian minus, it's the Hamiltonian over H bar omega, where is it? Yeah, its up there.

It's the Hamiltonian over H bar omega minus a half. So this is H over H bar omega minus a half, that's this thing and what's this? This is plus one I think, because the commutator the other way round was minus one so this is plus one. So that's that. H on N is equal to N plus a half H bar omega by definition. Right, so this on this produces N plus a half H bar omega. Divide by the H bar omega and we have N plus a half, take away a half, we have N, add one, we have N plus one. So this is in fact equal to N plus one.

So what follows this, so comparing this with this, we find that K2 is equal to N plus one. Going back to up there, we have that N plus one is equal to one over the square root of N plus one of A dagger N. This is a very important equation. If we would repeat, if we would go through this rigmarole the same logic using A on N being some other constant times N minus one, we would find that N minus one is equal to one over the square root of N of A operating on N.

We know that A operating on N is going to produce N minus one. It depletes the energy by H bar omega and therefore reduces N by one. The normalisation constant turns out to be one over the square root of N. I would recommend you check that after the lecture. It's a precise repeat of this logic here and it's worth remembering these rules, because this is such an important relationship. But they're very easy rules to remember. The normalisation constant is one over the square root of the biggest number that appears in the equation, right? So in this equation, the biggest number appearing is N plus one, that's what you use. In this equation, the biggest number that appears is N, that's what you use. So these are two very important equations which physicists remember.

Okay. So what we have to now. We're trying to find the wave functions fundamentally. This is just the tedious details of the normalisation, although those equations are of bigger use than just the wave functions. Let's find the ground state wave function. What is it? It's, we'll call it U zero of X and it is, of course, X nought. It's the function that is defined by this, the amplitude to be it X, if you were in the ground state, also called this, right, these are just two notations for the same thing. And this satisfies a natty equation because what we know, is that A operating on nought is equal to nought, right.

The ground state is defined to be, it comes into the world as the state, the one and only state that the operator, the destruction operator A kills stone dead. So if we multiply this equation on the left by X, we still have a valid equation. And this, and let's write in what A is. So A is M omega X plus I P. In principle it's over the square root of 2M H bar omega but because I'm about to put this stuff equal to nought, I can neglect the factor on the bottom. I can multiply through both sides of the equation, the factor on the bottom, and get rid of the garbage, right, clean things up.

Why don't we write this, I mean, we're more or less committed to writing this in the position, well, we have it there, right? This tells me that M omega X, X nought, because this is the position operator. This is an observable, I can imagine that it operates backwards on this. This is an Eigen function of that operator, so those are both operators. That's not an operator. This operates on this, if I want, backwards. This is its Eigen function, so I get an X, the number times this. This then meets that and produces this, which is our wave function. And then we also have plus I X P nought and the momentum operator was defined by saying that this thing is minus I D by the X of X.

This was the definition for any wave function of how P operated with an H bar, thank you, absolutely. So I need to write this stuff down, I need to write some more stuff down. This is equal to zero. This is equal to zero provided I put in the M omega X, X. If I rewrite that in wave function notation, this becomes M omega X U nought of X is equal to and let's clean this stuff up. Sorry not equal to, plus H bar D by the X of U nought of X equals nought. So what is this? This is a first order linear differential equation. And it's in fact an order differential equation because there are no other variables than X present, I've written as a partial derivative.

Consistency with what we're doing more in three dimensional cases I suppose, but it is a first order linear differential equation. There is no sort of differential equation that's more friendly, user friendly that we encounter. We solve such equations by using an integrating factor. Just to get this into standard form, I would actually write this as the u nought by the X plus M omega over H bar X U nought. This is the sort of standard form for a first order linear equation which you should remember from Professor Ross's course or whatever in the first year.

It has an integrating factor which is E to the integral of the coefficient of the linear of the constant term, well the not derived term, this term. E to the integral M omega over H bar X. The X which is clearly E to the M omega X2 over two H bar. That's its integrating factor and the equation is then, if you multiply by the integrating factor, the equation says that D by the X of the integrating factor times U nought is equal to the right side, which happens to be zero ergo this quantity here is a constant ergo U nought is equal to E to the minus M omega X2 over two H bar which I want to write as E to the minus X2 over four.

L2, why do I want to that? I want to do that because the probability associated with X, the probability of finding your particle of X, which is equal to U nought Mod2 will then be E to the minus X2 over two L2, so this is a normal distribution with dispersion L. So the reason I want to get this into this form, is in order, because I can identify that as the Gaussian width of the distribution, the width of the Gaussian distribution. Also once I say that this is the probability distribution, from my knowledge of statistics and stuff, I know what the normalising constant has to be.

I know that it's two Pi L2 root, two Pi L2 right, because that's the factor you need to normalise the Gaussian distribution. Oops I'm missing a minus sign, crucial, right, crucial. So what does L have to be, in order that this form is the same as that form? It's I think simple algebra to show that L has to be H bar over 2 M omega. L2 has to be that. Let's check that this has the right dimensions. One should always be checking one's dimensions in physics.

This has the dimensions and momentum time distance. So this has dimensions of M X P. So this is, sorry, this is M X V, which is what I should have said. Momentum times M V X. M V X is the dimensional structure here over . . .

[0 26:18 – 0:26:28 Break in Audio]

. . . dimensions here, so indeed this thing has dimensions of length squared, so L is indeed the length so it's okay. So what have learnt? We've learnt that the ground state of a harmonic oscillator has a wave function which is a Gaussian with a characteristic width given by this. And crucially, what we've already studied right, we've already studied distributions of particles which are Gaussian. Have wave functions which square up to Gaussians. And we now know what the amplitude distribution would be, or the probability distribution is for momentum from what we did before. We will have that the probability of the momentum will also be a Gaussian.

It will be E to the minus P2 over two, shall we call it sigma P2 over some square root two Pi sigma P2. And remember we had the uncertainty principle which said that the dispersion in X times the dispersion in momentum is equal to H bar over two for the Gaussians. This was a result we established when talking about the free particle. So that tells me that sigma P is equal to H bar over two L. So there's some characteristic width, there's some characteristic momentum that the particle has. So in the ground state, our particle is not stationary, it is moving, with a characteristic amount of energy.

It's also not at the bottom of the potential well, it has a characteristic amount of average potential energy. So we've come to the conclusion, we have an example here of one of the most important, perhaps the most important prediction of quantum mechanics, which is the existence of zero point energy. The basic issue is, that if we were to try to minimise the energy of the particle, which clearly ground state by definition does minimise the energy of the particle, is the state of the lowest energy by definition. If we're trying to minimise this energy, we want to get the potential energy to be as small as possible.

That clearly means moving towards the origin, getting as close towards the origin as you can. But there is, because of the uncertainty relationship because the more narrowly confined you are in real space, the more uncertain your momentum has to be. If you restrict yourself too much in position, to be too much at the bottom of the potential well, you will have a larger uncertainty in your momentum and you'll have kinetic energy. So in practice, in the ground state, there's a compromise between reasonably close to the origin and having a reasonably small kinetic energy.

So because of the uncertainty relation, quantum mechanical systems in their lowest states, have a finite extent spatially, even though really it's a point particle, but because there's a finite extent in which you'll find the particle, and a finite kinetic energy. And this is a totally, I hope you see that this is a specific example of a totally general phenomenon we have to expect to occur always, when we are considering particles trapped in some kind of potential well. And this is enormously important, because it's exactly this physics we will see. It is exactly this physics which determines the size of atoms, electrons.

So atoms here, are typically pretty close to their, near enough in their ground states and the size of these atoms is determined by the electron. If the atom gets any smaller and indeed, you know, you take a piece of steel or something and you stuff it into a press and squeeze the thing down, it will get smaller. But it will resist violently, you'll have to do work, you'll have to increase its energy to make it smaller and what happens is, that in order to make it smaller in real state, you have to give it more kinetic energy by the uncertainty principle and that's the work that you do, squashing it down. And you can see that idea worked out quantitively in one of the later chapters of the book.

So the size of atoms is determined by this zero point energy business and the uncertainty principle. And interestingly, the mass of protons and neutrons is not entirely, but it's overwhelmingly accounted for, by the kinetic energy of the quarks and gluons inside there, which are moving relativistically. They're in a very deep potential well and because, and very narrowly confined right, into ten to the minus fifteen metres. In order to be confined, even though they're fairly massive particles, into this very small space, they have to have a lot of kinetic energy. And the mass associated with that kinetic energy accounts for most of the mass, you know, of us. That's what it mostly is.

So this zero point energy phenomenon is extremely general, enormously important. And here we have the simplest, the classical example. In fact, you see, if we say H is equal to one over two M of P2 plus M2 omega2, X2 and we put in the uncertainty relation. We say either that X2, if we say that X2 is, sorry, if we say that P2 is equal to H bar2 over 2X. Let's get this right, sorry, over four because I've squared it. Over 4X2 sorry. So for the ground state, X2 is essentially the uncertainty in X, so it's associated with the uncertainty momentum in the same way.

If you stuff that into this, you put this relationship in. you find that H is one over two M is going to be whatever it is, H bar2 over four X2 plus M2, omega2, X2. This now is a function, maybe I shouldn't call it X2, maybe we should call it L actually, perhaps that would be more helpful. So I'm saying that X2 is on the order of L2 and P2 is on the order of this. So here we have a function of L and the minimum. If you ask yourself what's the, what value of L does this function have a minimum, the answer is it's that value that we gave up there for the quantum mechanics. So it's really true that L has been chosen to minimise the energy given the constraints imposed by the uncertainty principle.

Okay, so now we've found the ground state wave function, it would now be useful I guess, to show how we should calculate the, sorry, my notes are a bit out of order here. Yep, so what we want to do. Let's get the first excited state as an example. So first excited state wave function. So U1 of X which is by definition X1. What do we know? We know that 1 is equal to one over the square root of one times A dagger working on the ground state. So if I bra through by X, that tells me that U1 of X, which is equal to X1 is equal to, sorry the one over the square root of 1 doesn't need to be written any more.

One over the square root of N plus one and N here is nought, of A dagger, which is N omega X minus I P yes, over the square root of two M H bar omega. Now what we want to do is, it's helpful actually, to find out, is to rewrite this in terms of L, this characteristic length there. So let's just say what A dagger is in terms of L. Two M H bar omega, well where's, so if you take that equation up there that defines L and you multiply both sides by the square root of two M omega and then you multiply through by a square root of H, you find that, I need to write this down.

So L is equal to the square root of H bar over two M omega, multiply through by this, I find that the square root of two M omega is equal to the square root of H over L. If I multiply through by H bar, that's bar by the way, sorry. I multiply through by H bar, I find the square root of two M H bar omega, is in fact equal to H over L. So this factor here is equal to H over L, so I can say that A dagger is equal to M omega X. On the bottom is H over L, so an L here and an H bar there, minus I, I need, I have an H bar on the bottom and an L on the top times P.

Let's keep working. This stuff is looking remarkably like L all over again right. If I would multiply this two on the top and two on the bottom, then this would become one over L2. So this is equal to, the L2 would cancel this and I would have that this was X over two L, minus. Now let's put this in the position representation. In a position representation, this is minus I H bar D by the X. So the I's get together and make a minus sign and the minuses cancel each other, so we have an overall minus sign. The H bars cancel and this becomes L D by the X.

A dagger was billed as being dimensionless. Is it dimensionless? Yes, because we have an X over L and an L over X. So this is a handy formula for future reference. So let's find out what, sorry, I shouldn't have written complicated expression up there, it wasn't helpful, but U1 of X is equal to this baby, this animal, working on, what does it work on? It works on U zero of X, but what is U zero of X? Its business end is E to the minus X2 over four L2 and under here, I have to have a two Pi L2 to the quarter to power. This factor comes in, I said what the normalisation constant was. So let's see where that comes from.

I needed P of X to have this, so we obtained U as, I should have said a constant K times this, right. There was an arbitrary constant in this. Arbitrary constant of integration, which in fact, is going to be the normalising constant. So I know now that the wave function behaves like this, which gives me a probability that looks like this. The correct normalisation for the probability is this. So what I need to do now, is to say that in order to get things to work out well, I should replace that K with the quarter to power of what's inside here. So when you square it up, you find the right normalising constants for the probability.

So we've got the ground state now at last properly normalised. It has that quarter power and this, now this is going to come out, because we've paid proper attention normalisation. This will come out correctly normalised. And what happens is very simple. When we do this differentiation, we're going to bring down a minus X over two L2 and this L will cancel that and will be, cancel this, and will have an X over two L coming from here coming from here. We've got an X over two L coming from there, so the whole thing at the end of the day is one over this two Pi L2 one quarter of power. X over L, E to the minus X2 over four L2. That's the first excited state wave function.

To find the second excited state wave function, we'd use this self same operator. We're not going to do this, but let's just see what it would look like. U2 would be X over 2L minus L D by the X. One over two factorial, sorry one over the square root of two, that's the one over square root of N plus one operating on U1, which is X over L, E to the minus X2 over four L2 over two Pi L2 to one quarter of power. And you can see that what's going to happen is we're going to get an X2 term times use of the garbage.

We're going to get from this differentiation, we're going to get an X term from, we'll get various things, we're basically going to get an X2 term and when differentiating away, this will get a term in X to the nothing and when we bring this down, we'll get another X2 term as well, we'll multiply this and this will give us X2. So we're going to get terms in X2, in X to the nothing times E to the minus thingy. So this is going to be a poly, polynomial degree too. It goes by the name of a Hermite polynomial, it doesn't matter. And every time we use this operator, we're going to, we're going to get a more and more elaborate polynomial. Can you see that's, what's going to be the consequence?

Wave functions are all going to be this Gaussian that came with the ground state and then they're going to be times polynomials, which are going to be of order N. So the general state UN of X is going to be a Hermite polynomial H over X, E to the minus X2 over four L2, I'm not paying proper attention to the normalisation at the moment. And it's straightforward to find out what these are, you only have to differentiate, you don't have to do any clever, any things, just differentiate and they will all drop into your lap.

Something important to notice is that the ground state wave function is an even function of X, right. E to the minus X2, the first excited state wave function is an odd function of X because this operator is odd, right. It has one power of X in both places, as long as it changes side if you turn X to minus X. This one is going to be an even function of X, because we're going to apply another odd operator to a thing that's odd and we'll end up with an even result. So the ground state, well, ground state and is an even function of X. First excited is an odd function and the second excited.

So in fact UN of X is even if N is even and odd otherwise. We'll meet this phenomenon in other cases, but it's very often the case that the ground state is even and the first excited is odd and the next one is even and the next one is odd and so on, like that for similar reasons. And quantum mechanics has its own jargon for this. It says that this is an odd parity state, sorry even parity state. This is an even parity state. Parity just means is it even or is it odd, the wave function, and this is an even parity state. We'll have more to say about parity in a general context later on, when we're covering the material in chapter four.

Student Sorry are they both even parities?

Contributor Sorry, oh, no, this one is odd, excuse me. I was not sure which line I was on. This one is even. The ground state is even. The first excited state is odd and this is, it has a parity. We say this has a parity minus one to the N. So that jargon is used sometimes. I wouldn't worry about it. It turns out to be useful to know, we'll find it's useful to know whether your wave function is even or odd. It enables you to short circuit various computations. Okay. Ran out of board there. Let's have a go at this.

Let's work out the expectation value in the Nth excited state of X2. So we want, it would be nice now to build some connection to classical physics. Can we connect these results to classical physics? Classical physics, as I've said several times, is all about expectation values. The connection from quantum mechanics to classical physics occurs through expectation values. So let's work out this expectation value, which is going to enable us for example, to work out what the mean potential energy is, because the potential energy is proportional to X2 in the Nth excited state.

Now, how would we work this out? What we do, is we observe that A, now we had A dagger, we went to some trouble, here we go. Here is A dagger. A is going to X over two L plus I L over H bar P and A dagger we've got there, is X over two L minus I L P over H bar. So if you add these two equations, you discover that A times L, sorry L A plus A dagger is equal to X. So this is a very handy relationship. You express the X operator as a sum of annihilation and creation operators times L right. I've just added these two equations, the momenta have cancelled on each other. These have added up to give us an X over L.

Here we've had A plus A dagger, I've multiplied through by L. So when I want to work out this expectation value, I can replace each of those X's with this thing here. L2 comes out because it's only a number and I have N into A plus A dagger2 M. Let's multiply this out. It's L2 into N A2 plus A dagger2 plus A A dagger plus A dagger A. Now what's this? A2 applied to that, is proportional to N plus two, sorry N minus two, because each of theses A's takes away a unit of excitation. So A2 times that, is proportional to N minus two, but N minus two is orthogonal to that, so that makes no contribution to this expectation value.

Similarly, A dagger2 on this produces some multiple of N plus two, which is orthogonal to that, so that doesn't contribute. So these two terms don't contribute to the expectation value. These terms jolly well do contribute to the expectation value. We know that A on this, produces root N times N minus one and this produces root N, let me write this out. So this is going to be L2 of N A. A dagger on this is going to produce root N plus one of N plus one, right. This A dagger working on that, will produce root N plus one times N plus one. And then, and now I've left the other A to be done.

And here, we're going to say that is equal to N A dagger times the result of A working on that, which is root N times N minus one. Remember it's the square root of the largest integer occurring in the equation. And then this A is going to produce a root N plus one times N which will couple with this and produce one. So this is going to be L2, right, this is going to N plus one down to N which will produce a one when it meets this and we'll have another of these square roots. So we're going to have N plus one and when we use this on this, then this N minus one is going to be raised back to N which will produce a one when it meets this.

And we'll get another square root of N in the process, so we'll have plus N, so in other words, it's equal to two N plus one L2. So my time is up. What I want to do. So we've discovered something interesting which is the expectation value of X2 is two N plus one L2. We already knew it think about, did we already know about it? No we didn't in a certain sense. Yes we did. We knew it was for the ground state, N equals nought. We knew that it was a, we knew the probability distribution was a Gaussian and we knew that the dispersion of that Gaussian, i.e. the expectation value of X2, was L2.

We've now discovered how the dispersion increases when we add excitations and the probability distribution gets broader, but tomorrow, I want to connect this to classical physics.