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Contributor Yes, where we arrived with the harmonic oscillator yesterday was we had established that the expectation value of x squared in the nth excitated state was 2 n plus one l, squared. Yes of course, obviously on dimensional grounds, yes that's correct. So, and I said the next item on the agenda would be to connect that back to classical physics, always a valuable exercise, because it tells you something about quantum mechanics, it checks your results.

So of course classical physics doesn't know anything about 2 n plus one, the n the quantum number, the excitation number. But it does know about the energy and we know that n plus a half h-Bar omega is the energy, so we can write this as two times the energy over h-Bar omega. And this l squared, well l was defined to be h-Bar I think over 2 m omega, I probably had better check that my memory if that is correct, yes. So this l squared is an h-Bar over 2 m omega, so therefore this is the energy, various things cancel over omega squared, the twos cancel, oops we need an m, that survives in the algebra, is that correct?

So let's ask ourselves what do we expect, what do we expect classically? Classically we have that, what do we expect, we expect that the time average, well I'm going to write this down now, right, so the time average of x squared, which we'll call x squared bar since x is a simple harmonic function of time. The time average should be a half sorry this, this thing should be a half of x max squared, right? The average of cos squared is a half so if we are writing that x is equals to x cos omega t, it follows that x squared bar is a half of x squared, the maximum perturbation.

And what's the energy? The energy classically is a half k x squared, because when it's at maximum extension it has no kinetic energy, it only has potential energy, that's how much it has. Omega squared is root, omega is route k over m, so this is a half, so k, omega is the square root of k over m for a harmonic oscillator that being the spring constant. So if I want to get rid of k, I have to declare it to be omega squared, m squared, omega squared, x squared.

Right, so that leads to the conclusion that, I'm expecting that x squared is equal to 2 e over m squared. Omega squared, which is but this thing is equal to 2 of x squared bar, so this leads to the conclusion that x squared bar is equal to e over m squared, omega squared in perfect agreement with the quantum mechanical result. Oh dear, we need to put this up don't we?

I've gone adrift by a `[[?? 0:03:44]]`. Let me get rid off that stupid screen, I can only do one thing at a time, too stupid. Lights and blinds, screens, up, right screen, left screen, left screen, no, which side, who we talking about? You or me?

Okay. Right so there was a complaint. What went wrong, what went wrong was the squares of m, where did I goof on that. That was because e was a half, no that was correct.

Yes, that was beside the square root, rooty sign yes exactly, so this shouldn't have been there, so this shouldn't have been there then everybody's happy. Thank you.

Okay. So doing this check right of, what have we done, we have, we've checked that a bit, that classical, sorry. The quantum mechanical result agrees with the classical result. Now actually amazingly, we've been able to do this independent of n, right? In other words our classical physics or our quantum mechanics is recovered classical physics for all n. But we believe that we have to recover from quantum mechanics, classical physics only in the limit of large N because our classical experiments are all ones where we're moving macroscopic bodies around, where the excitation energy will be large in some natural units.

So we, the exercise that q m goes to classical physics for large N, large quantum number. Here's our first example of a quantum number, relevant quantum number, this is the correspondence principle. Correspondence and this is a, an early example in some senses, perhaps not a brilliant example because we get perfect agreement for all n right, but what we all, requiring is agreement for large N, but we really must have agreement for large N because classical physics is about, you know is being validated by experiments conducted at large N.

So, in the same, so let's talk about now the dynamics of oscillators. So far we have found these stationary states. And I've said several times these stationary states are highly artificial, one where you can see their artificialiality is the, that n t , the state with the energy n plus a half bar omega, at the time t is equal to that state at time t equals zero times e to the minus i, now this is e over h-Bar t, but since e is n plus a half h-Bar omega, this is n plus a half omega t.

So each and every one of these states has a phase which increments in time at a frequency n plus a half, n plus a half omega t but the oscillator, oscillates at a frequency omega, right? So we have to explain how it is that the oscillator oscillates at a frequency omega, but none of the states has a, evolves in time. None of these stationary states evolves in time with a frequency omega, not one.

So that, and moreover the oscillators that we are familiar with in the, you know in the school laboratory masses and weights and stuff, will have values of n which are like 10 to the, 10 to the 28 or 34 or that kind, simply gi-normous values of n so the frequency here will be stupendous. And nothing, and nothing in the laboratory is happening at that frequency, so this is totally, this is total fantasy land, we have to get back to reality.

We get back to reality by concentrating on expectation values because it's our connection to classical physics, which is what we call, is what we are pleased to call reality. So let's calculate the expectation values of x. If we do it for n we know we're going to get a constant, right? Because when we take the complex conjugate of this complex number, it will multiply together with the complex number over here and make 1. But we all really know this, we all really know that a stationary state has no time evolution what so ever.

So to get time evolution, what we need to do is, is say let's, the state of our system, we have to consider, to have something that moves we have to consider a system which does not have well-defined energy, which means that its wave function can, its state factor, wave function, whatever is a linear combination of states of well-defined energy. And let's suppose, let's take a simple example, let's suppose there are just 2, 2 states present, no sorry, the proposal is let's do a sum.

Let's do it in all generalities so we're going to write this as a n e to the minus i, e. So this is totally safe, any state could be written like this, so this is a side, the state of my system at time t. It's a linear combination of states of well defined energy. There's no question I can do that, it's a general initial condition.

And now let us work out the expectation value of x. So what is it? It's going to be the sum a n star, e to the I, n plus a half omega t times n, times x times m times a m, not starred e to the minus i, m plus a half omega t. So we can clean this stuff up to, this is a sum of n and m of course. It's going to be the sum n m a n star a m, e to the, when we put these two together we're going to have an e to the i, n minus m, omega t times n x m.

And yesterday we already saw what the stylish way is to handle this expectation value here, is to take advantage of an expression that we showed that the operator x can be written as l times a plus a dagger, where l is the thing we were discussing earlier on. It's a square root of h-Bar over 2 m omega, characteristic length. So, and we also saw what happened when we took an expectation value while we were doing a slightly harder problem yesterday. So this is going to be very straightforward now, because it's going to be l n a m plus n a dagger, all right?

And this, remember a on m produces m plus 1 in an amount the square root of m plus 1, so this is going to be l root m plus 1 of n m plus the square root, excuse me! This a produces m minus 1, whoops, m minus 1, sorry I'm not concentrating at all, right. Ah, but it's the square root of the bigger number, the biggest number that occurs, so this is the square root of a, sorry. a on m produces m minus 1, how much the square root of the biggest integer that's involved, that's the square root of m. And this is going to produce m plus 1, with the normalisation which is the squarer root of the biggest number involved, which is m plus 1, so n, m plus 1.

And what we want is, we have a here a sum of n and m, lets do the sum of m first, right, bearing in mind that that thing is this sum of a delta n, m minus 1 and a delta n, m plus 1. So we're going to have for our oscillator the expectation value of x is going to be, okay. Let's take this first one. While we do this first one sum over n, we're going to have that this is a n star, a how much. In order for this to be not zero, m has to be 1 bigger than n. So this is going to be the square root of n plus 1. And everywhere where I, everywhere where I see an m I'm going to have to write an n plus 1.

e to the I, and now in this case we've agreed that m is 1 bigger than n, so this is going to be e to the minus i omega t. S0 that's what we're going to get from this term and then sorry, sorry from this term when that goes in there. And now we have to put this in there, and now m is going to be 1 to get a non zero contribution, m is going to have, m plus 1 is going to have to be n, n so m is going to be n minus 1. So we're going to get an a n star a n minus 1 times the square root of n e. And now in this case, m is going to be smaller than n, so it's going to be easily plus omega t. And I've lost my l somewhere along the line. Let me re-instate it, right there is this l here, I hope we have everything.

Slightly scared that we haven't, let me just check that. No, it seems to be okay.

So, and we're still summing, I've lost a sum sign. We are still summing over n. Why don't we declare that in this, these are two separate sums and in this sum I can introduce a new notation, I can say that n primed is equal to sorry, n is equal to n primed minus 1, in this sum here. And then, sum of n primed and then I can re-label the n primed n and this term becomes the same as that term becomes the complex conjugative of that term. So when I do this, I'm going to have a sum now of n primed, l a primed, no not a primed, a n primed minus 1, a n that's starred, that star a n primed times the square root of n primed, e to the minus i omega t and the other sum is still over n and that's a star n a n minus 1 e to the i omega t.

But these, but n and m primed are the same, I mean they are just dummy indices, we're just summing over them, so this sum is in fact the complex conjugate of that sum. This two things are complex conjugates, so what we, if we, if we write a n a n minus, it is equal to say x n e to the i phi, which we can do where this is real and this of course is real too, because I'm writing a complex number, sorry that needs a star on it.

Then I'm going to be taking x e to the minus i omega t minus i phi plus this stuff e to the omega t plus i phi and we're able to combine the two exponentials, and discover that x is equal to l times the sum of x n cos omega t plus phi.

I'm sorry, we need a phi n on that, excuse me we need a phi n, so this obviously needs an n and that needs an m. Because each of these complex numbers has its own, has its own phase. So what have we discovered? We've discovered that lo and behold the position, the expectation value of the position does oscillate with periodically, this is now, this, we have indeed sinusoidal oscillation. At period 2 pi over omega, we have in fact recovered classical physics, the classical motion.

So the motion that this frequency occurs because of interference quantum interference between states. So these terms we've got are quantum interference between states of different energy, right. It was, why do we have states of different energy involved? It was because x, oh. It was because, so when we talked about the expectation value of x we got this huge long sum, which involved cross terms between states of it, it included the term n x n, right. That was also involved in here in which the same state of a given energy was present on both sides of x.

But that made no contribution to the sum because x is a sum that can be written as a sum of these ladder operators of these annihilation and creation operators. And if you put the same state at either side you get nothing. You only get something if the states on either side differ in energy by 1 unit of excitation .

So our result all arose from interference between states which differ in energy by 1, by 1 excitation. And it's a peculiar, so that's a very general phenomenon, and a peculiar feature of this problem is that those differences in energy are all the same, they're all h-Bar omega. And the frequency well, we'll see this in a moment, the frequencies, oscillations, so all these terms have the same sinusoidal. We have an infinite number of contributions still, but they all have the same sinusoidal behaviour.

So we've recovered the important feature of a harmonic oscillator that the, that the period is independent of the amplitude of the excitation . The amplitude of the excitation is controlled by which of these a n(s) are, are significantly large, right? Because a n is the amplitude to have energy n plus a half h-Bar omega. So a highly excited oscillator has, the non zero values of a n are all clustered at around the large value of n in a very, it's only gently excited oscillator has the a n(s) around, around zero or small values of n being fairly large. And therefore the sum will be, and this sum will be dominated by whatever region has the large values of a.

But the result we've got is that there's harmonic motion at frequency to, at frequency omega regardless of which terms in its sum are dominating. And that's this property that the period does not depend on the amplitude.

So let's, let's be more realistic and investigate, see how much more of classical physics we can get out of this by talking about an anharmonic, an anharmonic oscillator.

So I introduced harmonic oscillators by saying that they, they're widespread because if you have a point of equilibrium, if you plot against displacement from point of equilibrium you plot the force, you have some curve that looks like this and should pass through zero, and should pass through zero at the point of equilibrium, by definition of a point of equilibrium. Bt if you displace yourself from either side of the point of equilibrium, if it's stable the force slopes like this, it's positive, sorry, it really should be negative shouldn't it actually when I come to think of it. Sorry, I should draw the graph this way around shouldn't I, in order to get to the stable force.

So if I displace myself positively in x the force becomes negative and pushes me back. If I displace myself negatively the force becomes positively and pushes me back. So that's a stable equilibrium. And if in anharmonic oscillator arises, if we replace, if we approximate the curve, the curve of this force, versus distance by the straight line that's tangent to it at that point.

So basically what we are doing, so any, any force versus distance curve could be expanded as some kind of a Taylor series. And if we just take the first non trivial term in that Taylor series, we have a harmonic oscillator if we take subsequent terms, we will have a not harmonic oscillator, an anharmonic oscillator.

And typically the force versus, so for a harmonic oscillator the force versus distance is a straight line that goes all the way to infinity, which means that in order to pull your spring apart you have to do infinite work. Because to get x to go to infinity you have to overcome a force that goes to infinity, so infinite work is required to pull this thing apart, but all real oscillators, all macroscopic ones certainly you can just, you just break them. So they, only a finite amount of energy is required to push x off to infinity. And that's reflected in the fact that typically the force versus distance curve slopes over like this.

So that if we, if we plot the potential, v versus x in the harmonic case we have a parabola that looks like this and disappears off to infinity. So this is the harmonic oscillator. But in a real oscillator the force, the potential curve starts from some finite value as infinity, sorry and I should, I need to draw it so it becomes tangent to this and then disappears off like this. So this is a, this is a more realistic curve. An anharmonic oscillator is a good model if the parabola is tangent to the realistic curve over a decent range. That's the main idea.

So what we should do is investigate. Let's see what quantum mechanics has to say about more realistic oscillators. Let us take, so this is just an example, supposing we take v of x is minus some constant, a squared plus x squared. And I suppose we need an a squared on top to get the dimensions straight.

So supposing we take that to be our potential curve, then we can no longer, we can no longer, we now sit down, we have a perfectly well defined Hamiltonian p squared over 2 m plus this v of x, but we can no longer solve this analytically any more than we can actually analytically integrate the equations of motion, classically in this potential. So in either case you can't do it.

But it's pretty straight forward to solve this problem. h e equals e e numerically. We do it in the position representation, we bra through by x and have that x p squared over 2 m e plus x v e is equal to e x e, which turns into by the rules we've already discussed, turns into an ordinary differential equation minus h-Bar squared over 2 m d 2 u by d x squared plus v of x, u is equal to e u, where u of course is equal to x e.

So this is an ordinary differential equation, second order etc. And it's linear and it's pretty straight forward to solve numerically, if you look in the book there's a footnote that explains how to do that.

And I should have, I'm afraid, I meant to bring my laptop with the official figures. But when you do this so, so by discretising this differential equation, we turn it into a, an exercise in linear algebra which your computer solves. So you write this basically as a matrix, m on u which becomes a column vector the value that u takes at the different, at the different positions in x is equal to e u, so you turn it into a matrix equation and computers are very good at matrix equations.

When you do that, you discover what the values of e are, and you can also discover what these, what these wave functions look like and the crucial thing is that you find that if you plot the possible energies, you get distribution that looks like this. It starts off looking like an equally spaced ladder, for the harmonic oscillator there are steps there, each one of which is separated by h-Bar omega, for a n.

So we start off like that with the spacing given by the harmonic oscillator that's, tangent to the bottom of the curve, but as we go up the spacing gets less and less and less and less and less and less, and what essentially the algebra is doing is giving you an infinite number of allowed energies already in a finite range because this, is the nought. Okay, so that potential allows x, tht potential as x goes to infinity goes to a finite value my, well it goes to zero. This is zero and I guess this is minus v nought, so the lowest energy is somewhere down here.

So with only a finite range in energy, you pack in an infinite number of allowed energies, with a harmonic oscillator you have to, you pack in an infinite number of these things but in an infinite energy range, because this ladder goes on forever, right up to the heavens.

Okay. So that's the first, this is a very generic behaviour that we will encounter again in real systems. Now what's the physical consequence of that? Suppose we have, so now let's say our initial condition is this that it consists simply of two terms a n of n plus a n plus one of n plus one and so the time evolution is going to be this one's going, a, give myself some space, a n plus one, e to the minus i, a e n plus one t on h-Bar n plus one.

So that's, that's not a completely general condition now because I'm assuming that there are only two non vanishing amplitudes so my state, it happens to be such there are only two possible values of the energy that I can measure, there's an amplitude a n to measure the energy e n and there's an amplitude n plus one to measure the next highest energy. Okay, so this is kind of a special case.

If we now work out what the expectation value of x is for this special case, we find that it is err, a n star e to the i e n t h-Bar n, this is all very similar to the other case a n plus one star e i e n plus one t of h-Bar err, x and then the same stuff on the them a n e to the minus i e n t of h-Bar.

Now when we multiply this stuff out, we will get we will quite generally get only, two terms. We'll have this on this and this on this, the reason for that is that we will show later on, that for that potential x n so for the harmonic oscillator this is true. But it's not only true for a harmonic oscillator that this thing vanishes here, it's going to be true for any potential well which is symmetrical around x equals nought. So this follows from symmetry of v of x, that v of x is an even function of x, so long as v of x the potential is an even function, the same behaviour at minus x at plus x, this will vanish. We will show this as we go along, I haven't shown it yet but that will be true.

S, given that that's so, quite generally, my expectation value here is going to be a n star, so it's going to be this on this, e to the i e n minus e n plus one t on h-Bar, times the matrix element n x n plus one. Plus, oh excuse me, and I'm needing here some a n star a n plus one right? So that's that on that. And then we will have this on this a n a n plus one star, e to the minus i, e n minus e n plus one t on h-Bar times n plus one n x n. Have I done that right?

So what do we have now? We have again that this term is the complex, well we have this term is the complex conjugate, this term so we're looking at a sinusoidal function plus its complex conjugate therefore we're looking at something which is x n, could be written as x n cos e n cos e n minus e n plus one t on h-Bar plus a possible phase factor, all right? Where just to be concrete x n is the mod, is, is the modulus of a n a n plus one. So a n a plus one is the complex number, I have its modulus sticking here and here and I stuck its phase into there.

So what do we observe again? We have harmonic motion, sinusoidal motion. But look now at the period of this sinusoidal motion, the frequency of this sinusoidal motion now depends on n. Because again it's the difference of two energies of adjacent energies which counts, and as we increase n according to my bad sketch up there, the difference between adjacent energies gets smaller. So the frequency becomes smaller with increasing n. So we're recovering a classical fact which is that if you have made, if you have an anharmonic oscillator of a typical type, and you make bigger, you kick it to a bigger oscillations its period will slow down.

And that's true of an ordinary pendulum. An ordinary pendulum has its highest frequency if you have it go to and fro with a small amplitude that clock makers make their pendulums go to. But the period goes to, as you increase the amplitude of the oscillation so this is as it were, high, high omega for a pendulum. As you boost the period to the point at which it's about to go over top dead centre. You know if you make it swing so it goes like this and then like this, the period goes to infinity formally. Well, it does go to infinity it's hard to do experimentally as you increase the amplitude to the point of which it would go, just keep on going. It had enough energy to go right through top dead centre.

So, so this slowing of the period with the increasing amplitude is manifested in a standard pendulum and we see how it emerges from the structure. So here we're learning something important, that the way in which the dynamics is encoded in the spacing of the energy levels, of the stationary, the energies of the stationary states. These are just simple examples of what's totally generic.

Okay. Something else that we can learn about this anharmonic oscillator is, so more generally this was a simple example. I said, let's consider in order to get something to move, I considered a state with, that had undefined energy. to keep it simple I took just two non vanishing amplitudes to, only two, there were only two non vanishing amplitudes in the expansion of the state in stationary states.

Realistically we would have, if you take an ordinary pendulum like that and you, and you give it a jog you will, the energy will be uncertain by zillions of values of h-Bar omega and many, many, many, many of these coefficients will be non, will be non vanishing. So more generally, we're going to have that upsi is equal to, it will have many terms and let's just write down a few of these terms, a n minus one, e to the minus i, e n minus one, t on h-Bar plus a n, e to the i, e n etc. There will be a, so there will be many, many of these coefficients.

But if we know pretty much what the energy of the oscillator is, right, we've lifted our bob up to 30 degrees or something and let it go, the energy is not completely undetermined. And what that means is that many of these will be non zero, but they will all be clustered around some particular value of n. So that if you look at the, at the value of one of these amplitudes, the modulus of it, as a function of n, you'll find that you'll get a pattern sort of like this somehow, all right? There will be an n which the, at which the amplitudes peak and there will be small values here, because we're pretty certain the energy isn't that small. And small values here because we're pretty certain the energy is that large.

So that's the generic situation and where we come in and calculate the expectation value of x, what we now have is the same sort of thing as up there, but it's somewhat more complicated. We're going to have an n, a n star a n minus one, sort of the things that we had before, e to the i, e n minus e, e n minus one t on h-Bar times some matrix element n x n minus one. And then we will have, actually I wanted to do this the other way round didn't I?

Then I need to put in a minus sign there. Right, then I will have, the next one I will have because n x n will vanish, the next one I will, by this symmetry property I will have a n plus one star a n e to the minus i, e n minus e n plus one t on h-Bar, h-Bar times n x n plus one. And then I will have, not plus, it's equals but plus a n plus three star a n, it'll be the next term, a to the i, e n minus e n plus three t over h-Bar, n x n plus three plus dot, dot, dot, right. This is a specimen of a disgusting expression which would give us the expectation value of x.

This combination of terms we've already seen. This is nothing really new. The harmonic oscillator had just this kind of thing. In the case of a harmonic oscillator this energy difference was exactly minus this energy difference, making this exponential and this exponential complex conjugates. They will not now be exactly, this will not be exactly this because this is the difference between n and n minus one. And this is the sort of one step in the ladder, and this is the size of the step one above it on the ladder which would be slightly smaller.

Male I think you have your `[[?? 0:42:17]]` the wrong way round in that.

Contributor Oh thank you, yes I have. That's right because I changed my mind how I was going to do this didn't I? Thank you very much. So, this should be n plus one, and that should be n and this should be n plus three. Okay. It's good to know that there's understanding in the room.

So that's one thing that's going to happen because this is a more realistic oscillator, is these frequencies will be changing and this, crucially this frequency here will be present where it wasn't present in the harmonic oscillator case. In the harmonic oscillator case this number here vanished. In principle this would have been here but this matrix element vanishes for the harmonic oscillator. But it's not generally going to vanish; it's going to stick around.

Now that has an important consequence because this frequency, so e n plus three minus e n is going to be on the order of three times e n minus, e n minus one. Right, because it's the difference. It's three steps on the ladder and that's only one step on the ladder. And if we think of the size of the steps on the ladder becoming smaller gradually as we go up the ladder, which will be, is as good a picture to use, then this term is going to be essentially three times the frequency associated with the other two, which we can regard as about the same.

So when we assemble all this stuff, we're going to find that x, the expectation value of x looks like some number times cos of e n minus en plus one, no let's declare this to be three omega n, right. So we're defining omega n to be this quantity here and we're going to find that we have a cos, some term like cos omega n t, and we're going to have some other term with some other coefficient times cos three omega n t. And we're going to have some other term with x five, some other coefficient times cos five omega n t.

So that this number will depend on, will contain products of stuff like a n a n plus three star and a n plus three a n plus five star etc, right. And this will contain things like a n a n, plus, plus five. But we will have these other frequencies present and this is what leads to, so this series implies periodic motion but anharmonic motion.

So, in a musical instrument you, the note, the motion of the string in a piano or the vibrations in an organ tube or a flute tube or whatever, has its, it has a well-defined frequency which sets its pitch. But the particular tone of the instrument is determined by the characteristic numbers of higher harmonics which are present, because it's an anharmonic oscillation, typically.

But there's more that we can do here, which is connecting to classical physics, which is to make the point that, if we arrange this stuff, right. So this expectation value of x, we take out the leading term. We say that this is e to the minus i omega n t. And then we're going to have some sum, no let's, sorry, it's better, it's gets very complicated if you take out the exponential value of x. It's easier if we think about of psi itself as the function of time. We look at psi itself as a function of time, we can take an e to the minus i e n t on h-Bar out and we can say that this is dot, dot, dot a n n plus e to the i e n plus one, e n plus one t on h-Bar n plus one times a n, sorry. This needs to be in, yes, we need a minus sign there. I need a t on h-Bar there.

So, I've taken a common factor out, so this one doesn't have any exponential. This one should have its proper exponential minus the thing that I've taken out. The next one should have, and this should be a n plus one. This should be plus a n plus two e to the minus i, e n plus two minus e n t over h-Bar n plus two, and so on and so forth.

So to a lowest order of approximation these differences here are all multiples of a common frequency. And after a period so when, e n t over h-Bar is equal to 2 pi these things, sorry. n plus one minus e n, after the time it takes for this to come round to 2 pi, this will have come round to 4 pi almost, and so on and so forth. And so the wave function will look, all this sum will be the same as it was at t equals nought, because all of these exponentials would have come round to one again.

That's in the case that these things are all multiples of the same frequency. But as we've seen they're not quite multiples of the same frequency. This is slightly smaller than twice this. And so in the time it takes for this one to come round to 2 pi, this one has come round, isn't quite round to 2 pi. And even more so further down the line, and therefore the wave function isn't quite back to where it was when, at t equals nought. And as we, as this, as time goes on and we count more periods so this becomes 2 pi n, these discrepancies become more and more and more important. And these terms down here, when this one has come round to 2 n pi, this one will be significantly off 2 n pi, and this one even more so.

And that means that the wave function is not returning to its original value. And we're looking at motion which is not periodic. And whereas initially, because we'd released our particle from some particular point in the potential well, these wave functions all constructively interfered here at a particular value of x. After a certain number of basic periods the interference, the constructive interference here and the destructive interference everywhere else will become less and less, exact. And the distribution of our particle will become more and more vague, until after a very long period. The phases of these will be essentially random and we'll have no knowledge of where it is.

And this is precisely mirrored in classical physics. In classical physics the small uncertainty in energy that was associated with having more than one a n in this series, was associated with a small uncertainty in period. If the period, if the energy was very high the period would be very long and after a long time the particle would have gone around and around a million times and a million and a bit times. And here it would be. But if it was slightly different, a slightly lower energy it would have a slightly higher frequency, and it would have done a million and one oscillations and it would be over here.

So that you can see the smaller uncertainty in energy is going to lead after a long time through the small uncertainty in period to a large uncertainty and phase in a total scrambling of our prediction of where it is.

So, again quantum mechanics is returning in a rather complicated way in through quantum interference, a result that we're very familiar with if we think about the classical situation. It's time to stop.