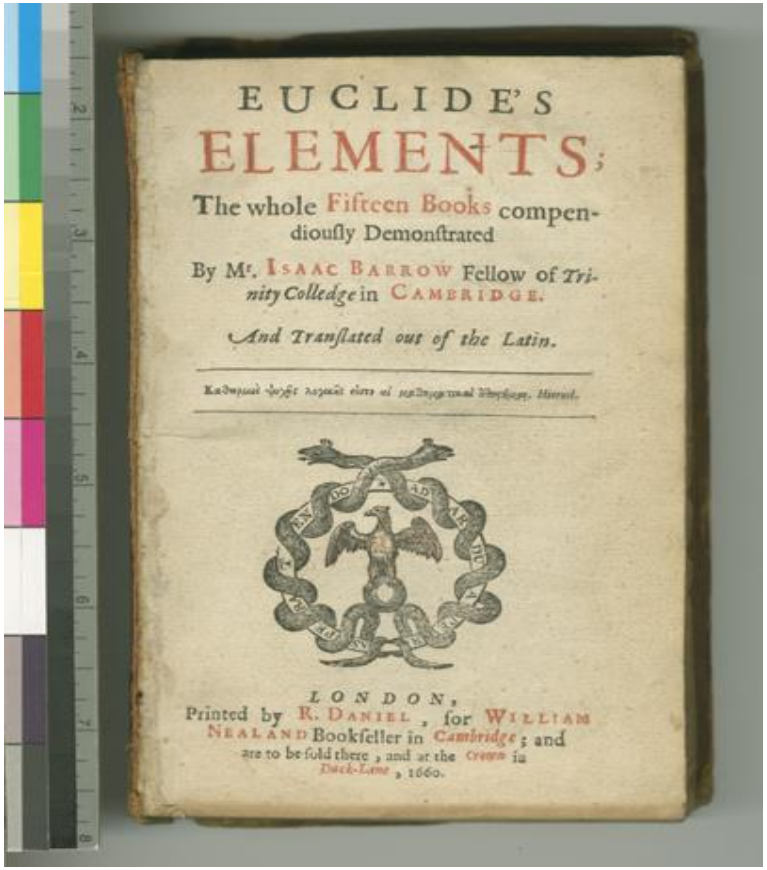
Mathematics and Culture: Geometry and Everything Else

Judith V. Grabiner
Pitzer College, Claremont, California
Oxford, 10 December 2015



EUCLIDE'S

ELEMENTS;

The whole Fifteen Books compendiously Demonstrated

By Mr. ISAAC BARROW Fellow of *Trinity Colledge* in CAMBRIDGE.

And Translated out of the Latin.

Καθαρμοὶ ψυχῆς λογικαὶ εἰσιν αἱ μαθηματικαὶ ἐπιστήμαι. *Hierocl.*

LONDON,
Printed by R. DANIEL, for WILLIAM NEALAND Bookseller in *Cambridge*; and are to be sold there, and at the *Crown* in *Duck-Lane*, 1660.

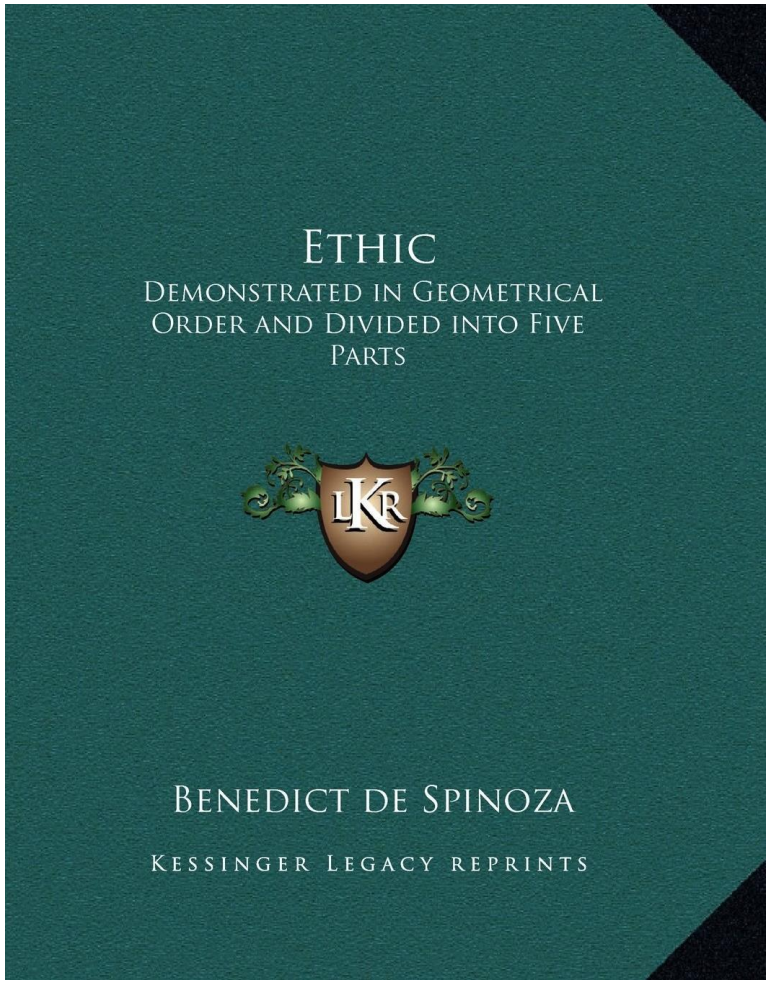
Euclid
(325–265 BC)

Plato on the left, Aristotle on the right, from Raphael's painting "The School of Athens."

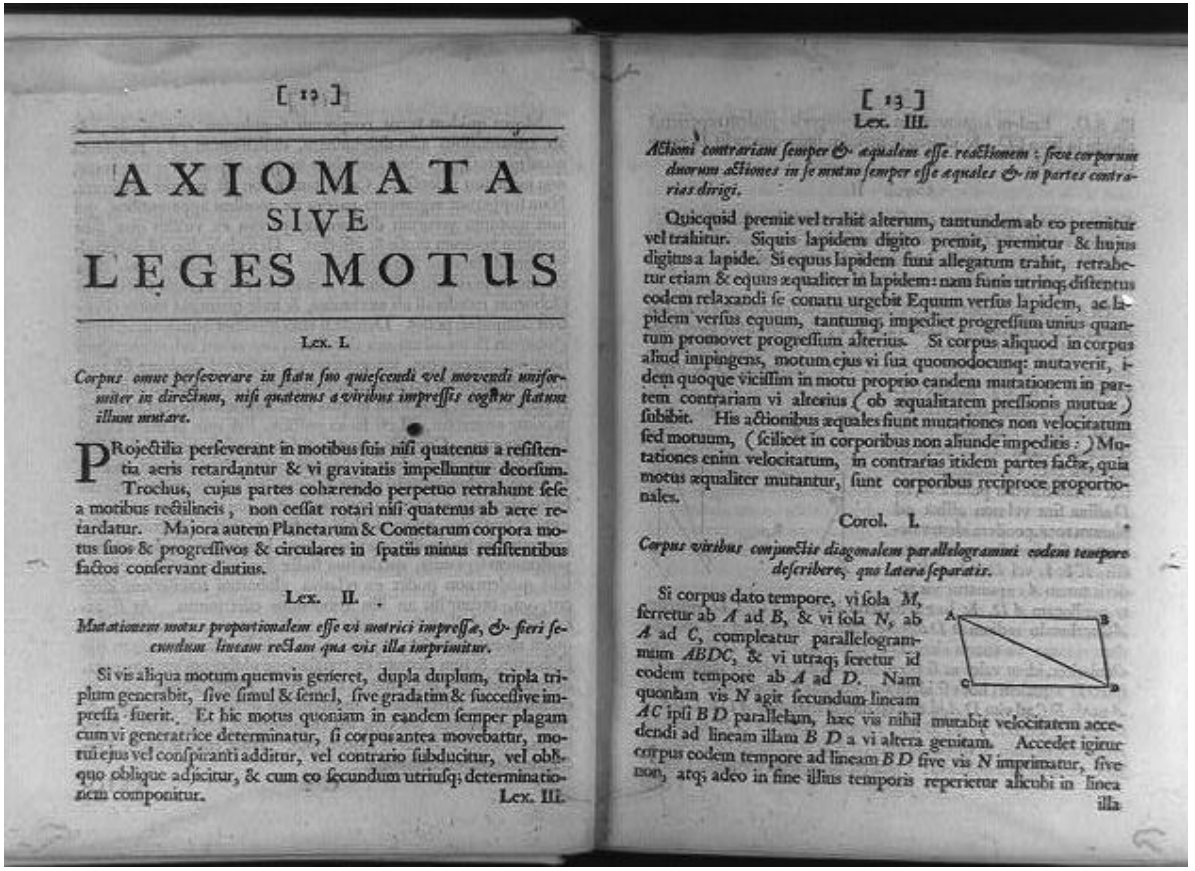
René Descartes: If we start with self-evident truths and then proceed by logically deducing more and more complex truths from these, "**there cannot be any propositions so abstruse that we cannot prove them, or so hidden that we cannot discover them**."

Discourse on Method, 1637.

Benedict Spinoza (1632 – 1677)



From Newton's *Principia* (1687)



[12]

AXIOMATA
SIVE
LEGES MOTUS

Lex. I.

Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare.

Projectilia perseverant in motibus suis nisi quatenus a resistentia aeris retardantur & vi gravitatis impelluntur deorsum. Trochus, cujus partes cohærendo perpetuo retrahunt sese a motibus rectilineis, non cessat rotari nisi quatenus ab aere retardatur. Majora autem Planetarum & Cometarum corpora motus suos & progressivos & circulares in spatiis minus resistentibus factos conservant diutius.

Lex. II.

Mutationem motus proportionalem esse vi motrici impressæ, & fieri secundum lineam rectam qua vis illa imprimitur.

Si vis aliqua motum quemvis generet, dupla duplum, tripla triplum generabit, sive simul & semel, sive gradatim & successive impressa fuerit. Et hic motus quoniam in eandem semper plagam cum vi generatrice determinatur, si corpus antea movebatur, motui ejus vel conspiranti additur, vel contrario subducitur, vel obliquo obliquè adjicitur, & cum eo secundum utriusq; determinationem componitur. Lex. III.

[13]

Lex. III.

Actioni contrariam semper & æqualem esse reactionem: sive corporum duorum actiones in se mutuo semper esse æquales & in partes contrarias dirigi.

Quicquid premit vel trahit alterum, tantundem ab eo premitur vel trahitur. Siquis lapidem digito premit, premitur & hujus digitus a lapide. Si equus lapidem funi allegatum trahit, retrahetur etiam & equus æqualiter in lapidem: nam funis utrinq; distentus eodem relaxandi se conatu urgebit Equum versus lapidem, ac lapidem versus equum, tantumq; impediet progressum unius quantum promovet progressum alterius. Si corpus aliquod in corpus aliud impingens, motum ejus vi sua quomodocunq; mutaverit, idem quoque vicissim in motu proprio eandem mutationem in partem contrariam vi alterius (ob æqualitatem pressionis mutuæ) subibit. His actionibus æquales fiunt mutationes non velocitatum sed motuum, (scilicet in corporibus non aliunde impeditis:) Mutationes enim velocitatum, in contrarias itidem partes factæ, quia motus æqualiter mutantur, sunt corporibus reciproce proportionales.

Corol. I.

Corpus viribus conjunctis diagonalem parallelogrammi eodem tempore describere, quo latera separatim.

Si corpus dato tempore, vi sola *M*, ferretur ab *A* ad *B*, & vi sola *N*, ab *A* ad *C*, compleatur parallelogrammum *ABDC*, & vi utraq; feretur id eodem tempore ab *A* ad *D*. Nam quoniam vis *N* agit secundum lineam *AC* ipsi *BD* parallelam, hæc vis nihil mutabit velocitatem accedendi ad lineam illam *BD* a vi altera genitam. Accedet igitur corpus eodem tempore ad lineam *BD* sive vis *N* imprimatur, sive non, atq; adeo in fine illius temporis reperietur alicubi in linea illa

We hold these truths to be self-evident,

that all men are created equal…

That whenever any Form of Government becomes destructive of these ends, **it is the Right of the People** … **to institute new Government**…

The history of the present King of Great Britain … repeated injuries and usurpations, all having in direct object the establishment of an absolute Tyranny …

To **prove** this, let facts be submitted to a candid world.

…

We, ***therefore***, the Representatives of the United States of America…solemnly publish and declare, that these United Colonies are, and of Right ought to be Free and Independent States.

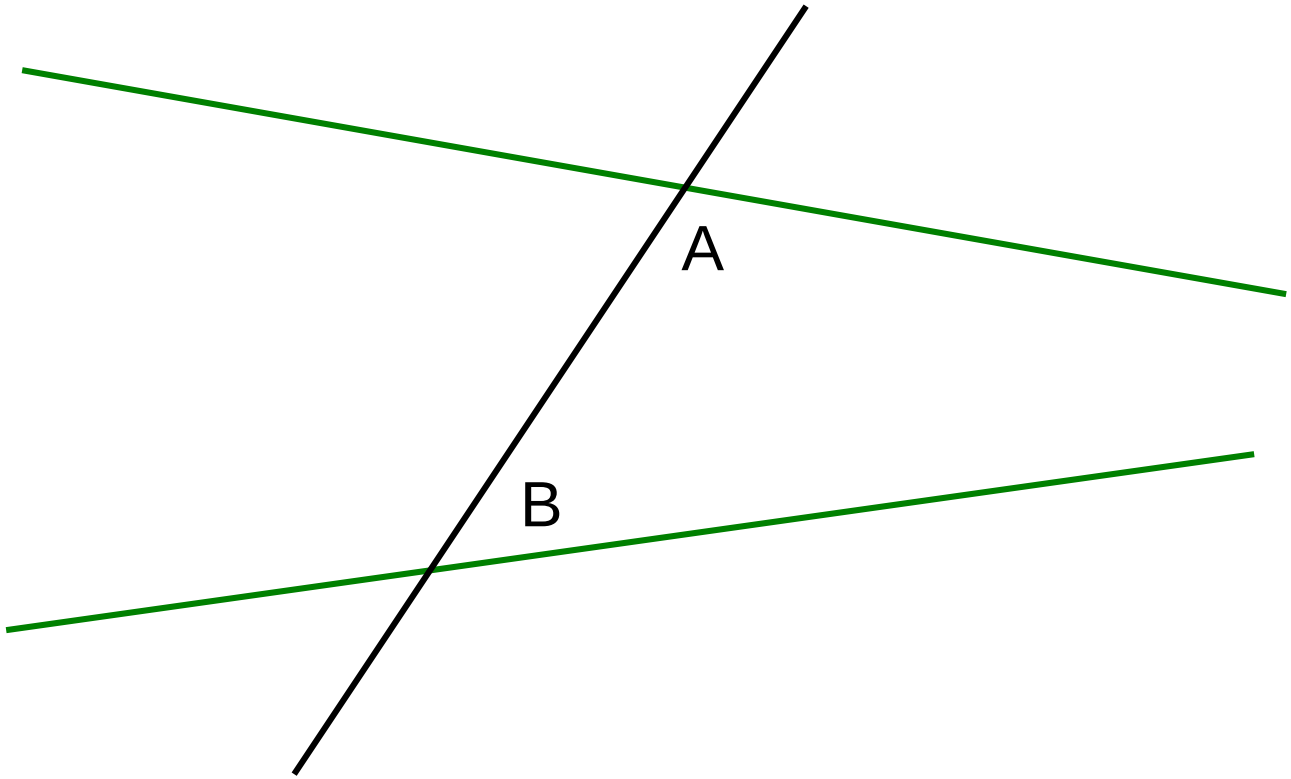
Euclid's 5 postulates

1. A straight line can be drawn from any point to any other point.
2. A finite straight line can be extended as long as you want.
3. A circle can be constructed with any point as its center and with any length as its radius.
4. All right angles are equal to one another.

5. (The so-called "parallel postulate")

   If a straight line that falls on two straight lines makes the interior angles on the same side add up to less than two right angles, then the two straight lines, if produced indefinitely, meet on that side.

Euclid's Fifth Postulate says: If angle A plus angle B add to less than 2 right angles, then the green lines must meet!



Some medieval Islamic and Jewish mathematicians who worked on Euclid's parallel postulate:

Ibn al-Haytham (965-1039)

Omar al-Khayyam (1048-1131)

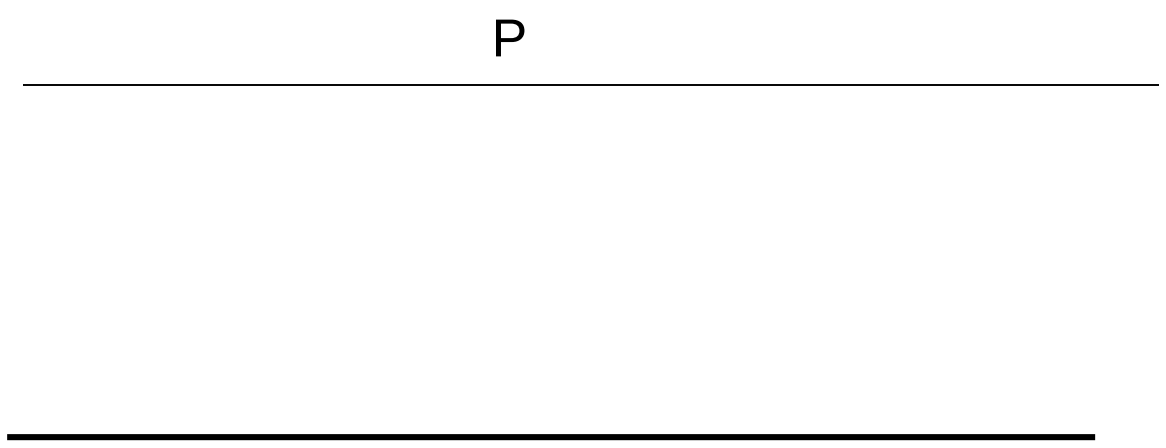
Nasir al-Din al-Tusi (1201-1274)

Levi ben Gerson (1288-1344)

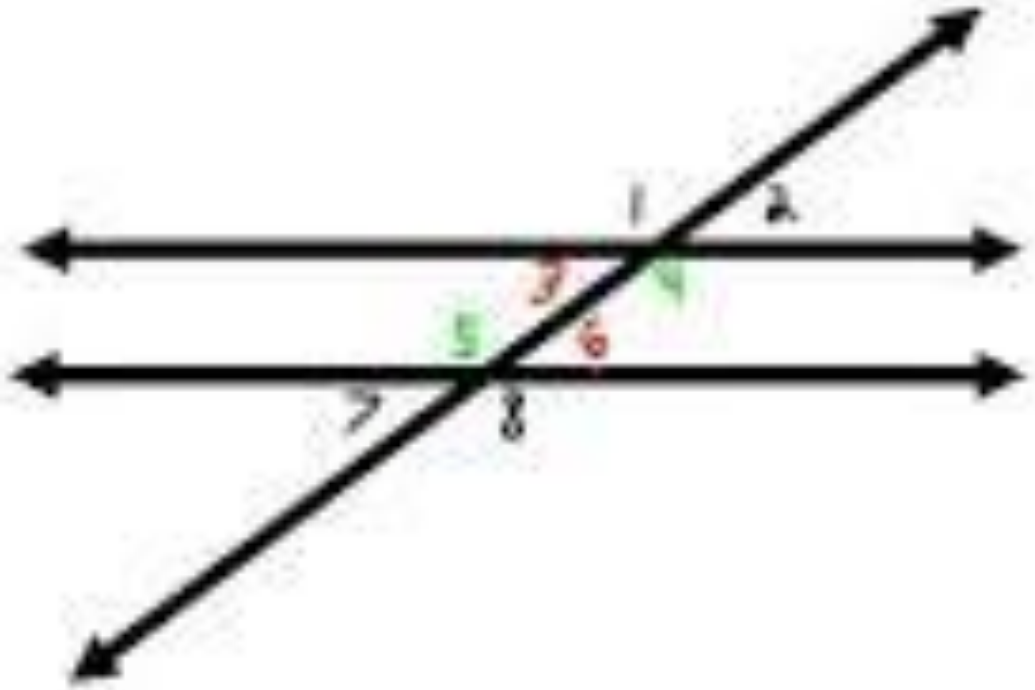
Some European mathematicians of the 17th and 18th centuries who worked on trying to prove the Fifth Postulate (plus al-Tusi, whose work Wallis & Saccheri knew):

John Wallis, Girolamo Saccheri, Johann Heinrich Lambert, Adrien-Marie Legendre, Joseph-Louis Lagrange

There is only one parallel to the bottom line through the point **P**.



If two parallel lines are cut by a 3d line, the alternate interior, and the corresponding, angles are equal (*Elements*, Book I, Prop. 29)



The Principle of Sufficient Reason

- For everything that is, there's a reason why happens as it does and not otherwise.

Lever, with equal weights at equal distances from the fulcrum, must balance, says Archimedes. **Why?**

Lever, with equal weights at equal distances from the fulcrum, must balance, says Archimedes. Why?

Well, why not? (The principle of sufficient reason!)

Giordano Bruno (1548 – 1600)

Gottfried Wilhelm Leibniz (1646 – 1716)

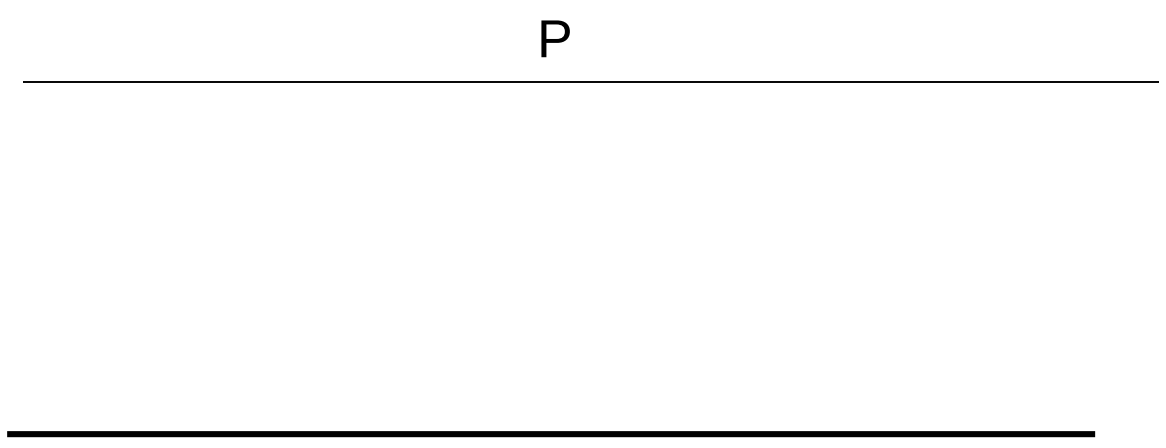
Newton's First Law: Every body perseveres in its state of being **at rest** or of **moving in a straight line at a constant speed**, unless a force acts upon it.

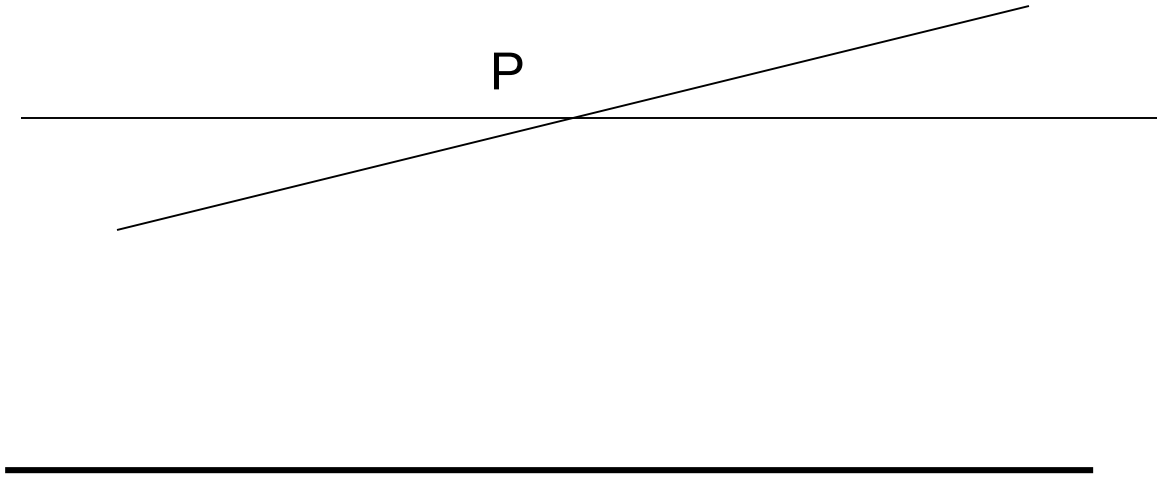
Joseph-Louis Lagrange (1736-1813)

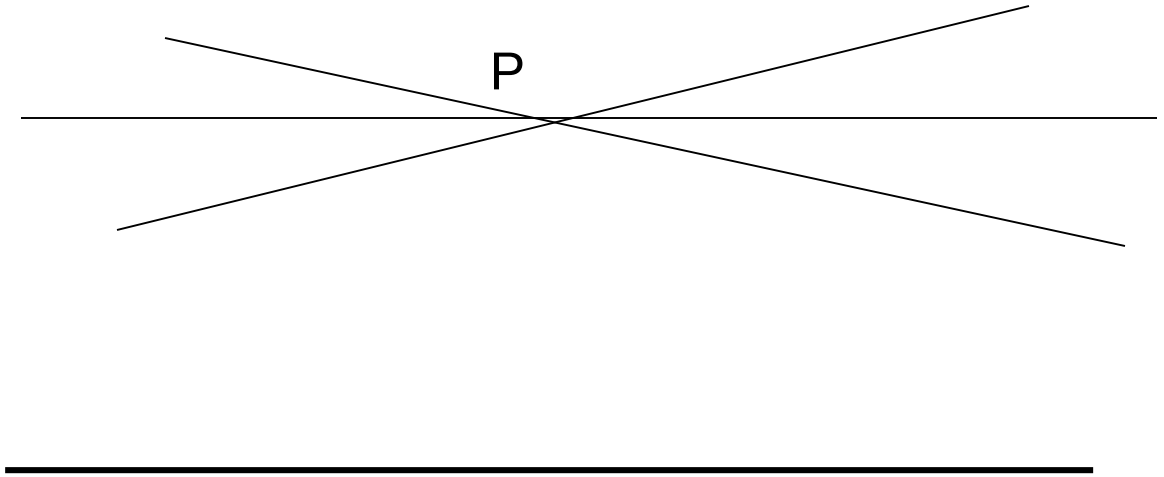
Europeans honor their scientists on stamps

(plus you get his dates for free)

Given a line and an outside point P, there is only one line through point P that is parallel to the original line.
(That's the postulate found in most high-school textbooks.
It is logically equivalent to Euclid's Postulate 5)




P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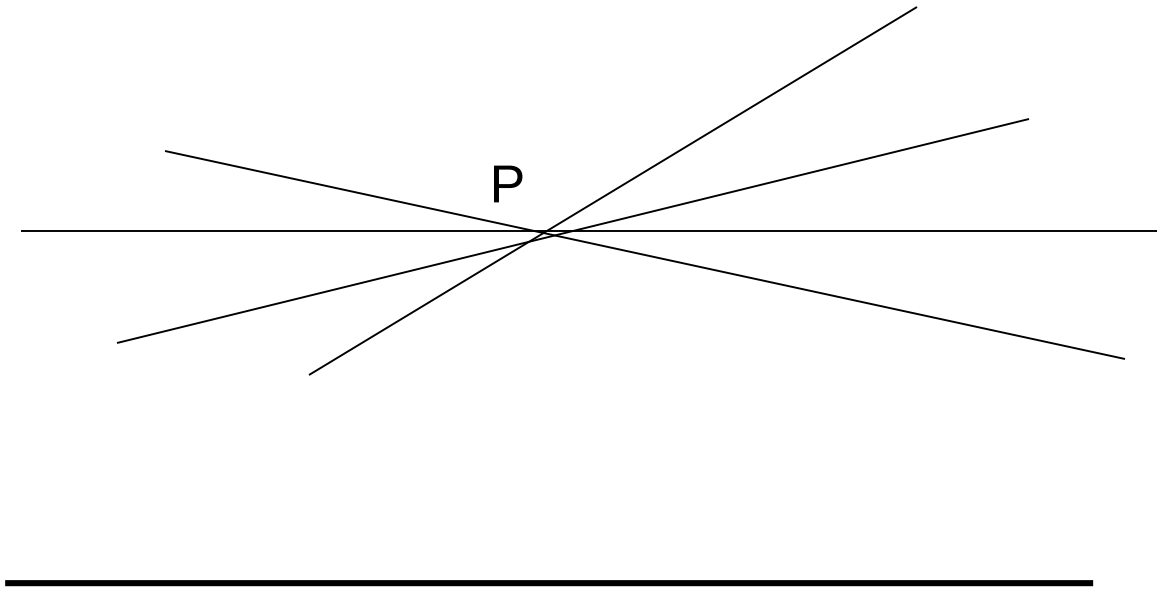
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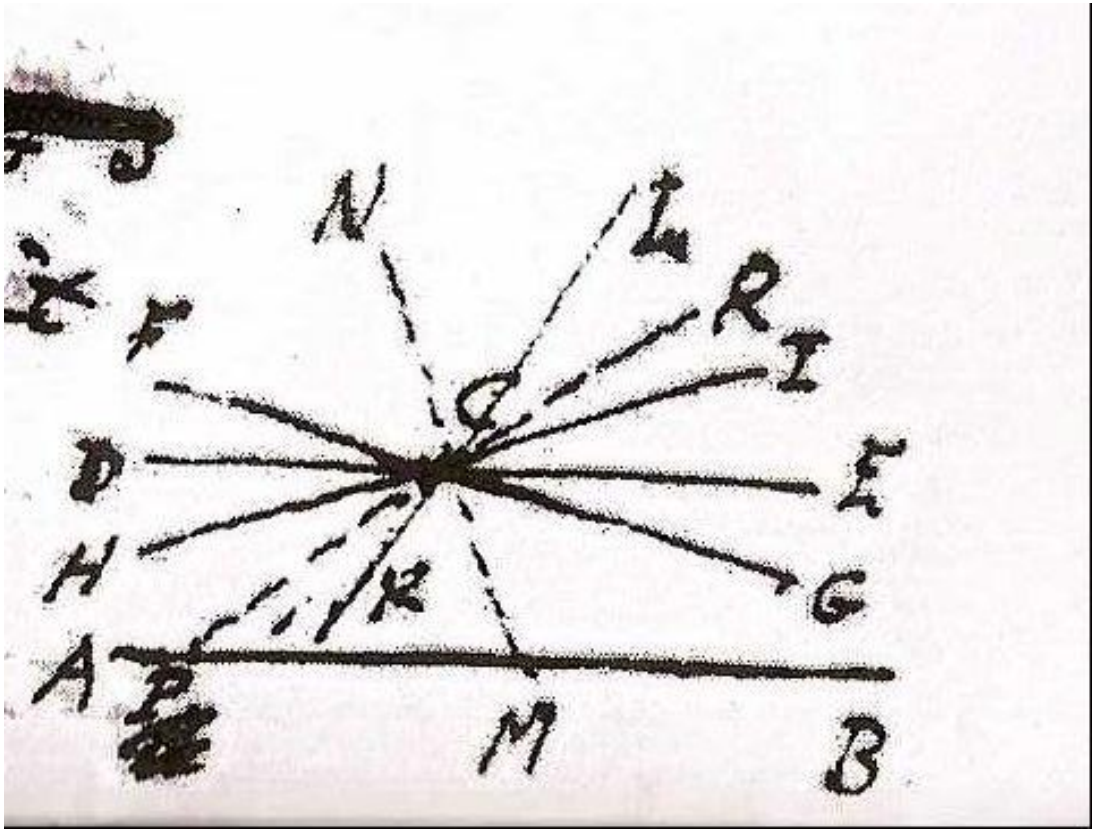
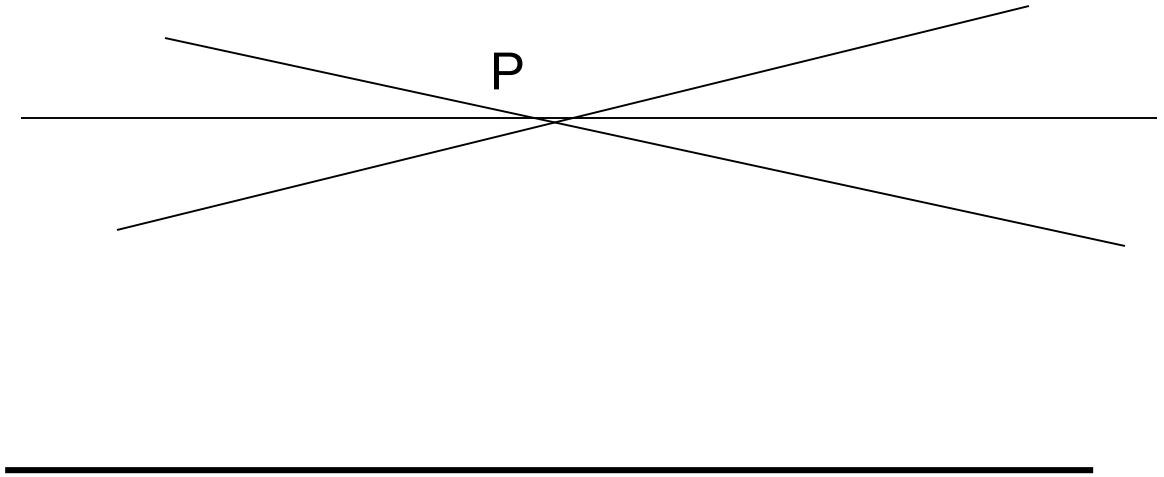
Lagrange's diagram (1806)

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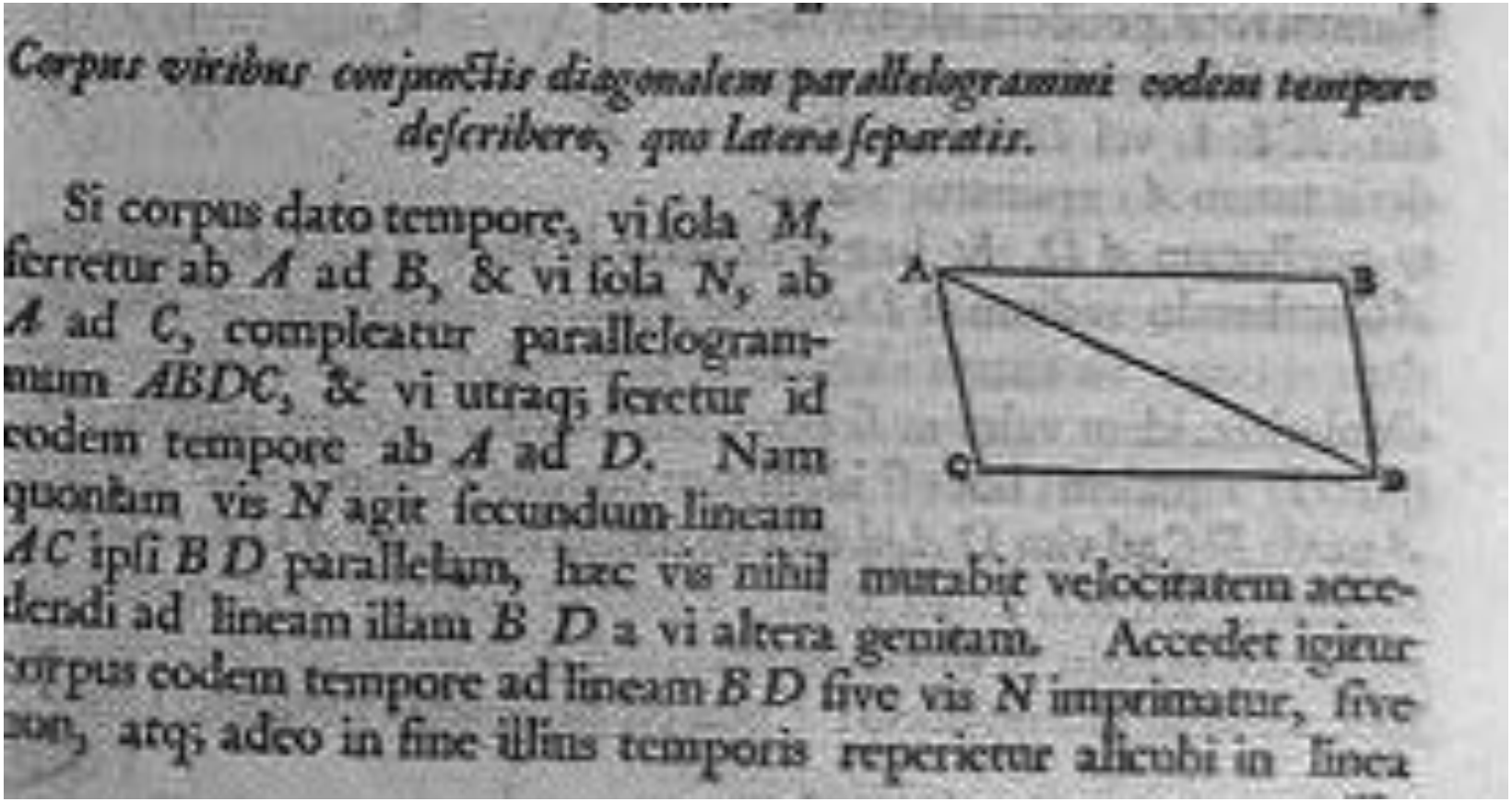
From Newton's *Principia*:

A body acted on by two forces conjoined will describe the diagonal of a parallelogram in the same time that it would describe the sides if acted on by those forces separately.



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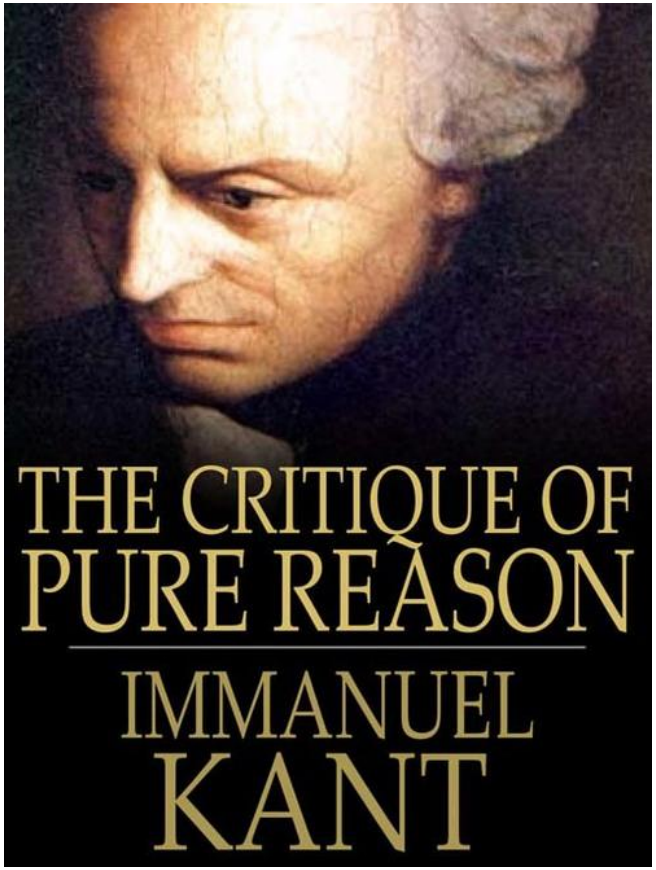
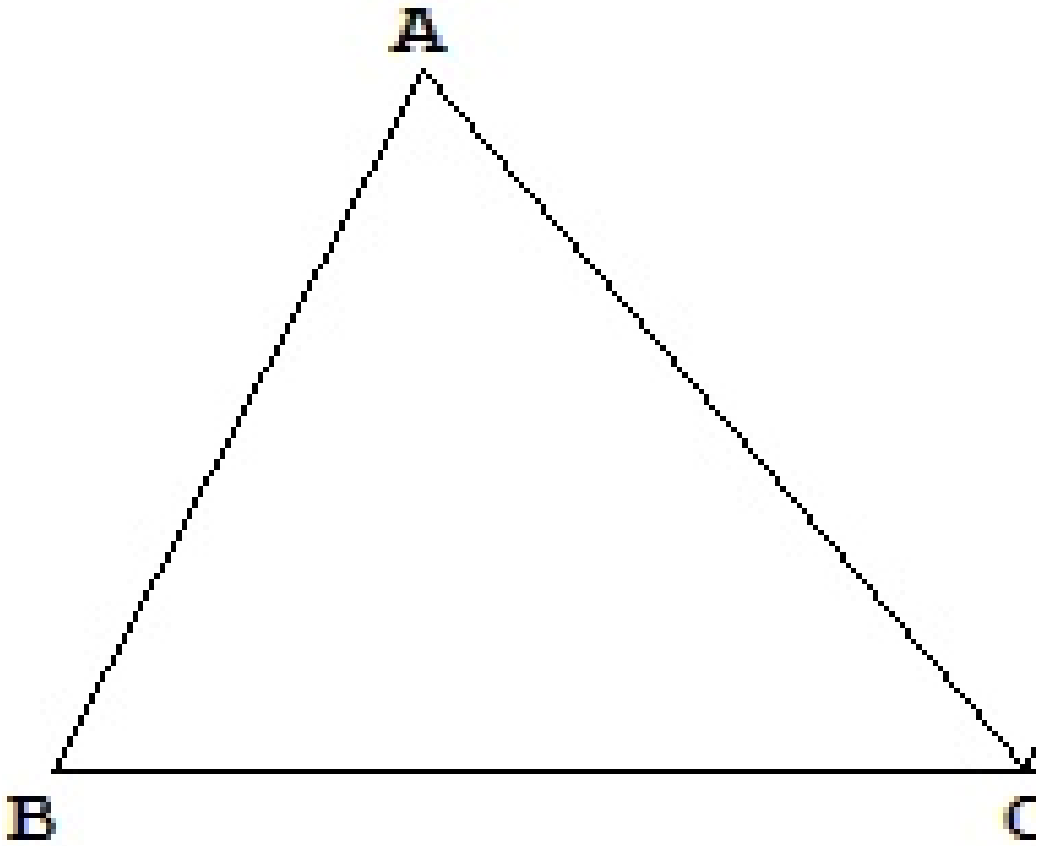
A B C D

Leonhard Euler (1707 – 1783)

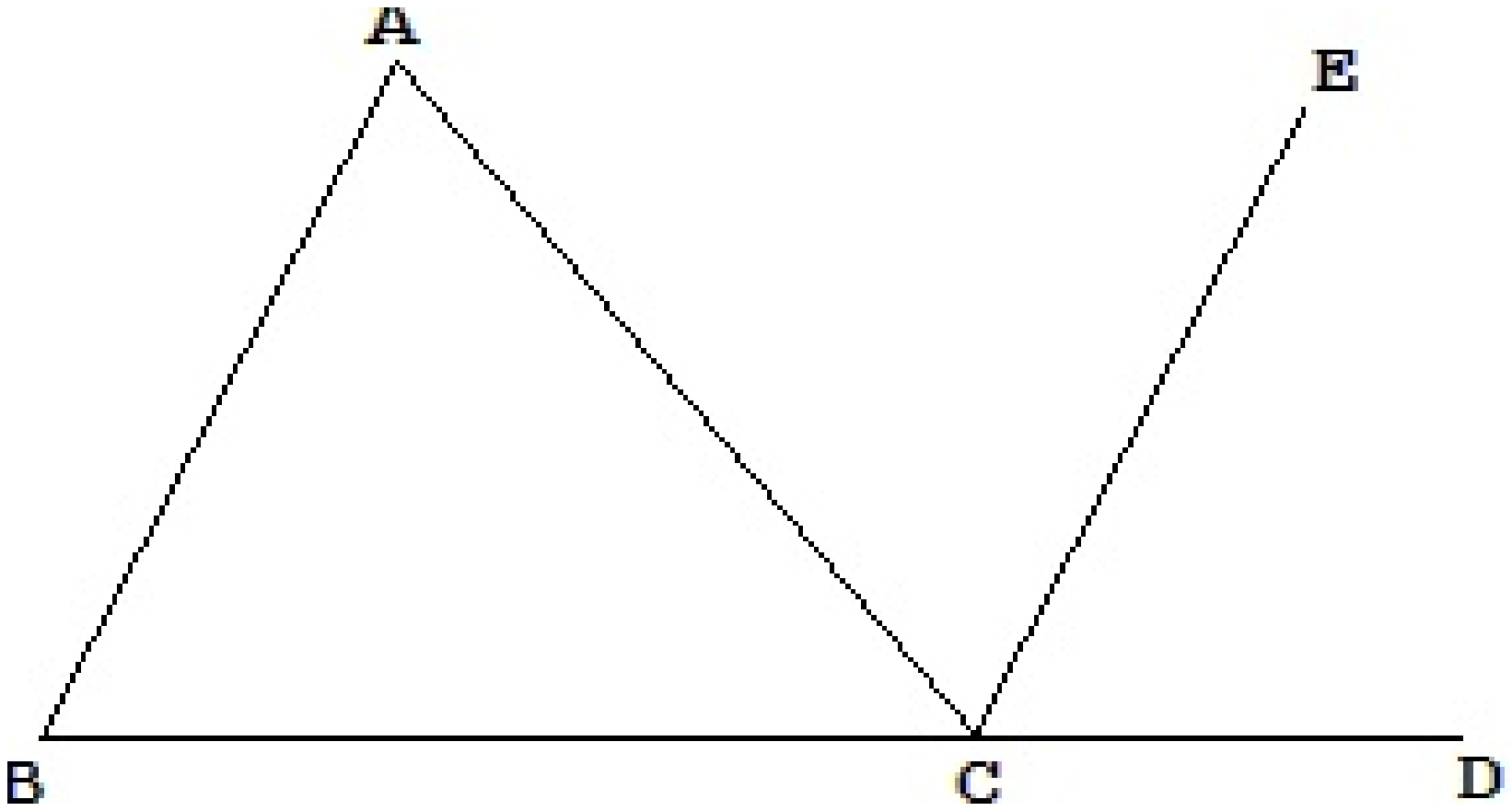
Immanuel Kant (1724 – 1804)



Given triangle ABC. Let's prove angles **A+B+C = two right angles** (from Euclid's *Elements*, Book I, Theorem 32).



Extend the line BC to D, and construct CE parallel to AB



Since AB is parallel to EC,

< A = <ACE (alternate interior angles)

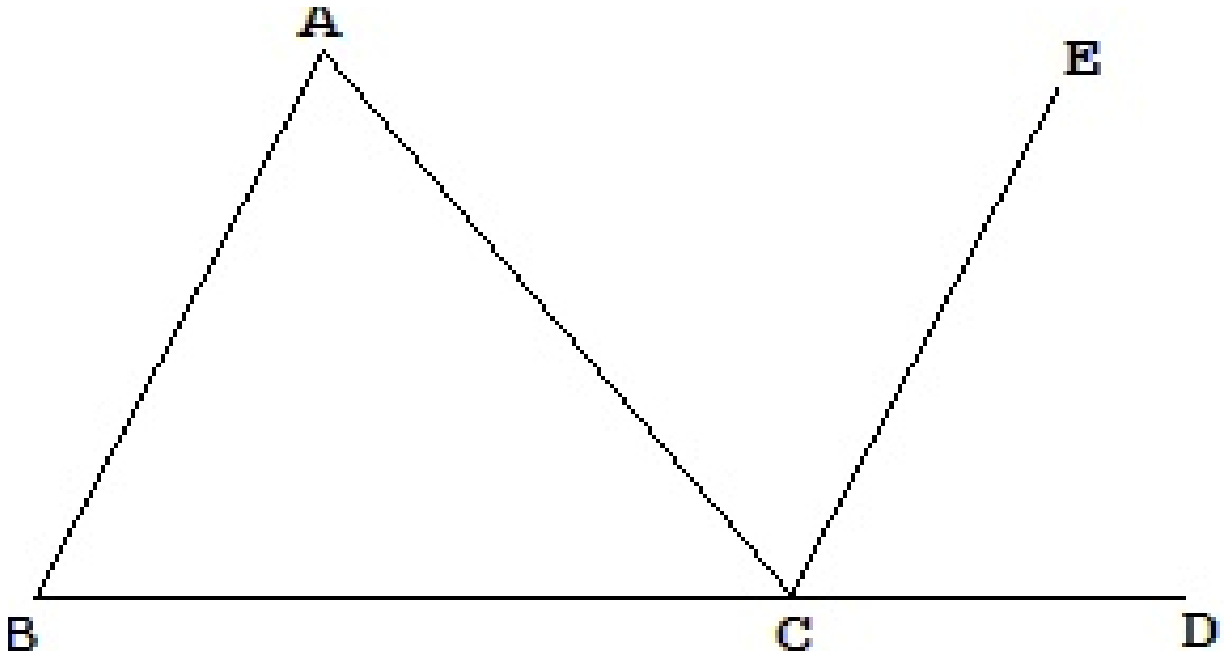
< B = <ECD (corresponding angles)

<ACB = <ACB (everyone knows THAT).

Add these equations up!

Right-hand side, the 3 angles around C, add up to two right angles because they're on a straight line.

Therefore the 3 angles of the triangle do so as well.



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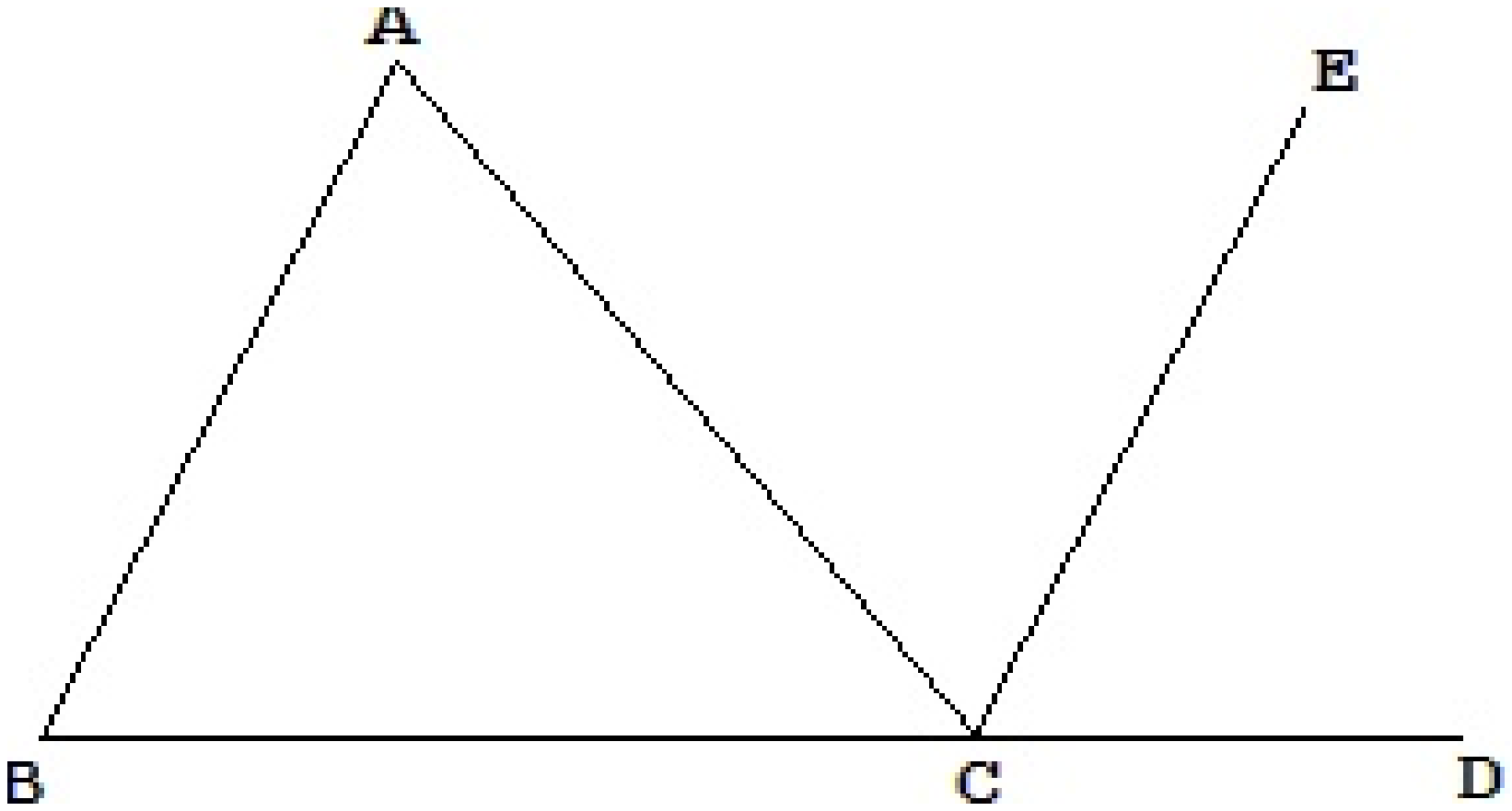

Since **AB is parallel to EC**, previous theorem implies:

< A = <ACE **(alternate interior angles equal)**

< B = <ECD (**corresponding angles equal**)

<ACB = <ACB (everyone knows THAT).

Add these equations up!

Right-hand side, the 3 angles around C, add up to two right angles because they're on a straight line.

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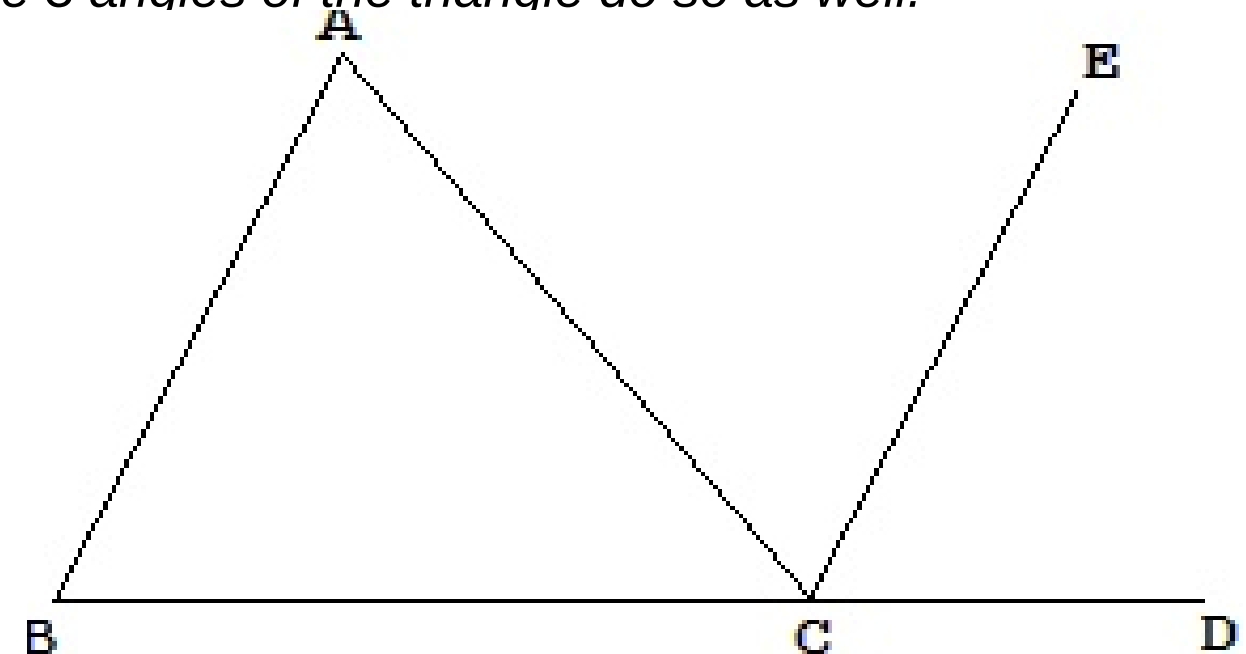
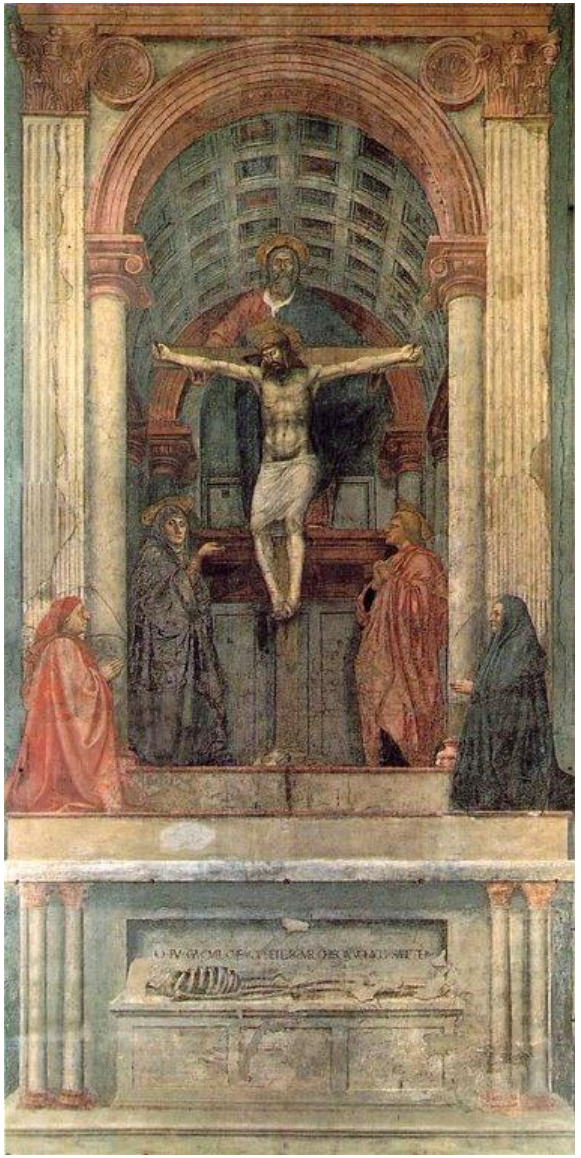


Voltaire (1694 – 1778)

Masaccio, “The Trinity” (1428)

Giorgio Vasari said in 1550 of this pioneering painting, “It looks as though the wall is pierced.”

Photographs do this for us; Renaissance artists needed sophisticated geometry.



The Bayeux Tapestry (11th century)

Nicholas of Verdun, “The Crossing of the Red Sea” (c. 1181)

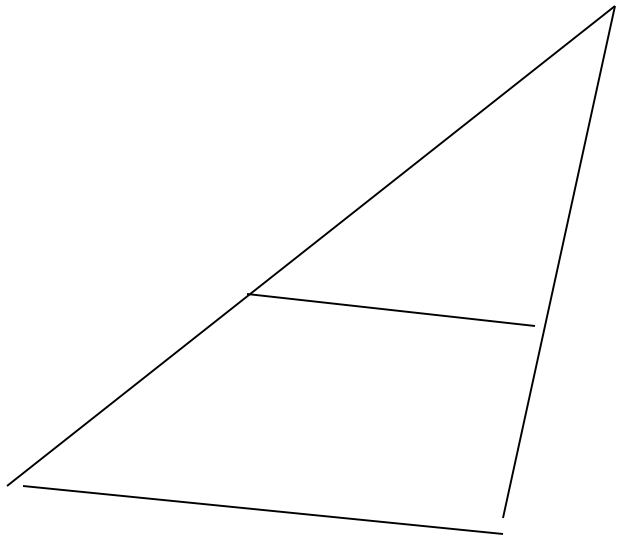
Piero della Francesco or Luciano Laurana, “The Ideal City,” c. 1470

Euclid, Optics, Proposition VI:
Parallel lines seen from a distance appear to be an unequal distance apart

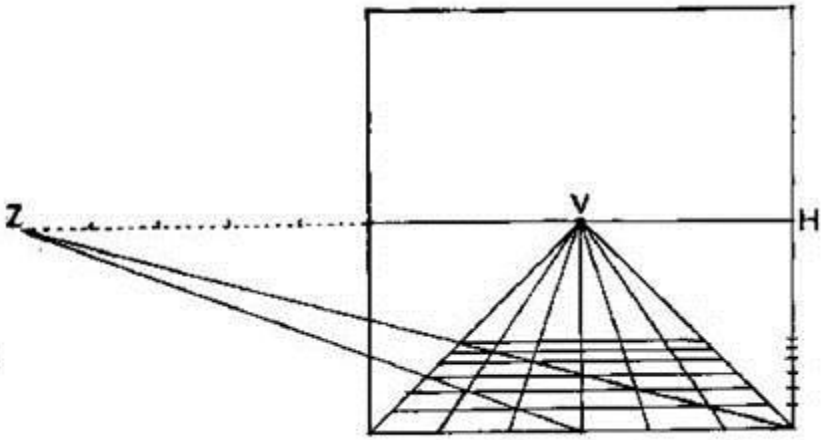
“I’ll tell you EXACTLY how far apart.”

Euclid, Book VI, Theorem 2:

If a straight line is drawn parallel to one of the sides of a triangle, then it cuts the other two sides of the triangle proportionally

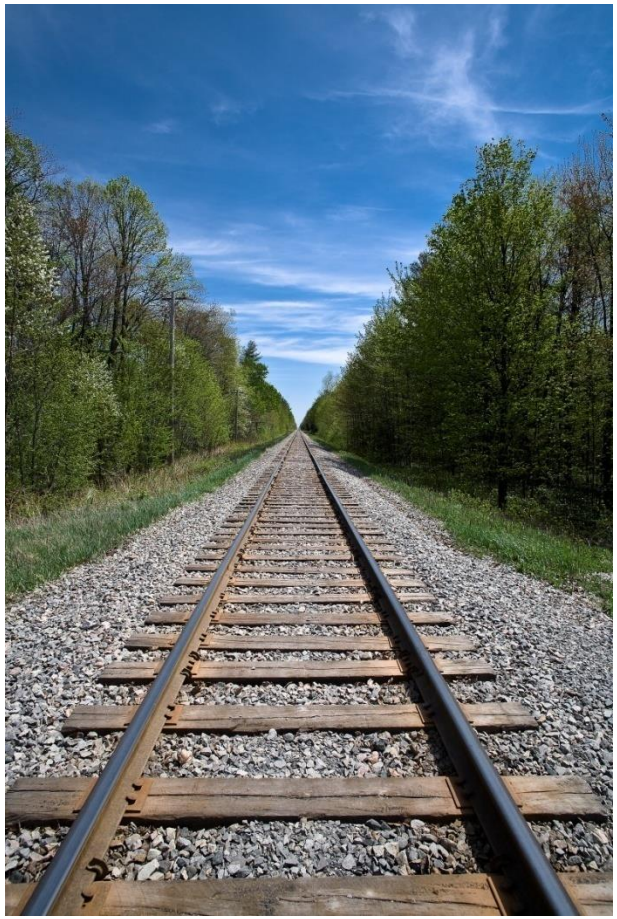


Alberti Construction (after Leon Battista Alberti, 15th century)



Parallel horizontal lines perpendicular to the plane of the picture are drawn to meet at a point V, the principal vanishing point.

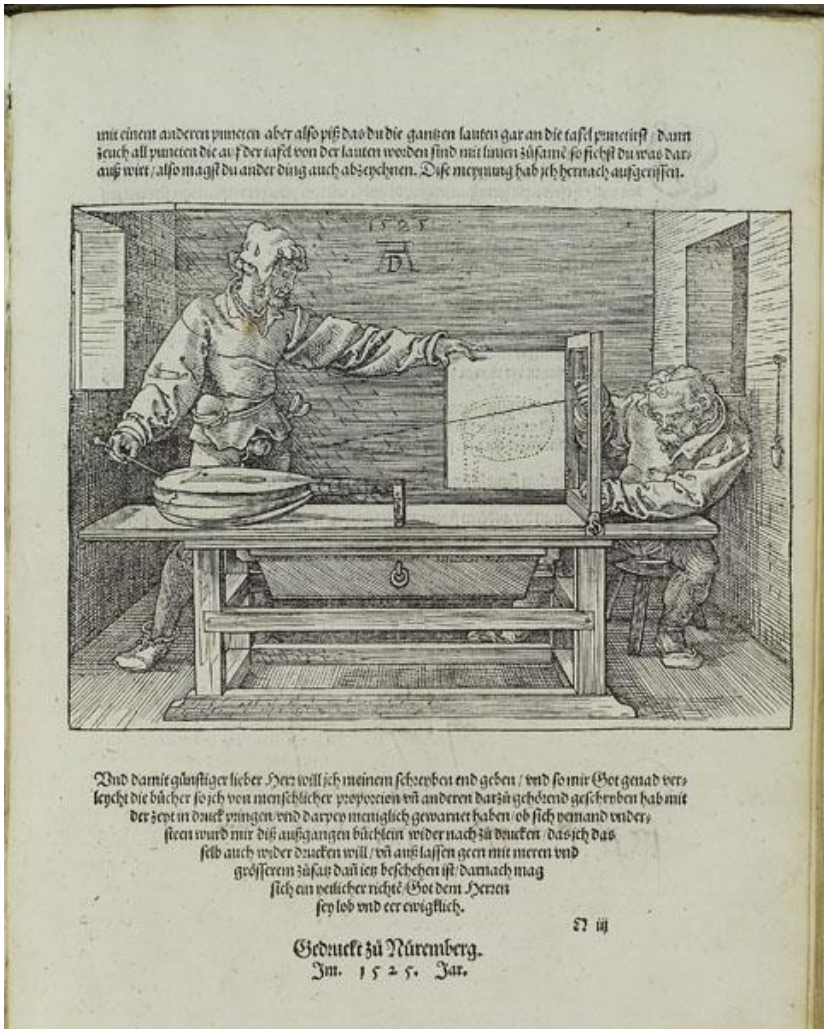
Parallel horizontal lines perpendicular to the plane of the picture are
drawn to meet at a point V, the principal vanishing point



Piero della Francesca, “Flagellation of Christ” 15th century

Leonardo da Vinci, “The Last Supper” (c. 1498)

Albrecht Dürer
How to draw a lute in perspective (1525)



mit einem anderen punctten aber also piß das du die gantzen lauten gar an die tafel punctirst / dann zeuch all punctten die auf der tafel von der lauten werden sind mit lauten zůsamē so siehst du was darauß wirt / also magst du ander ding auch abzeychnen. Dise meynung hab ich hernach außgerissen.

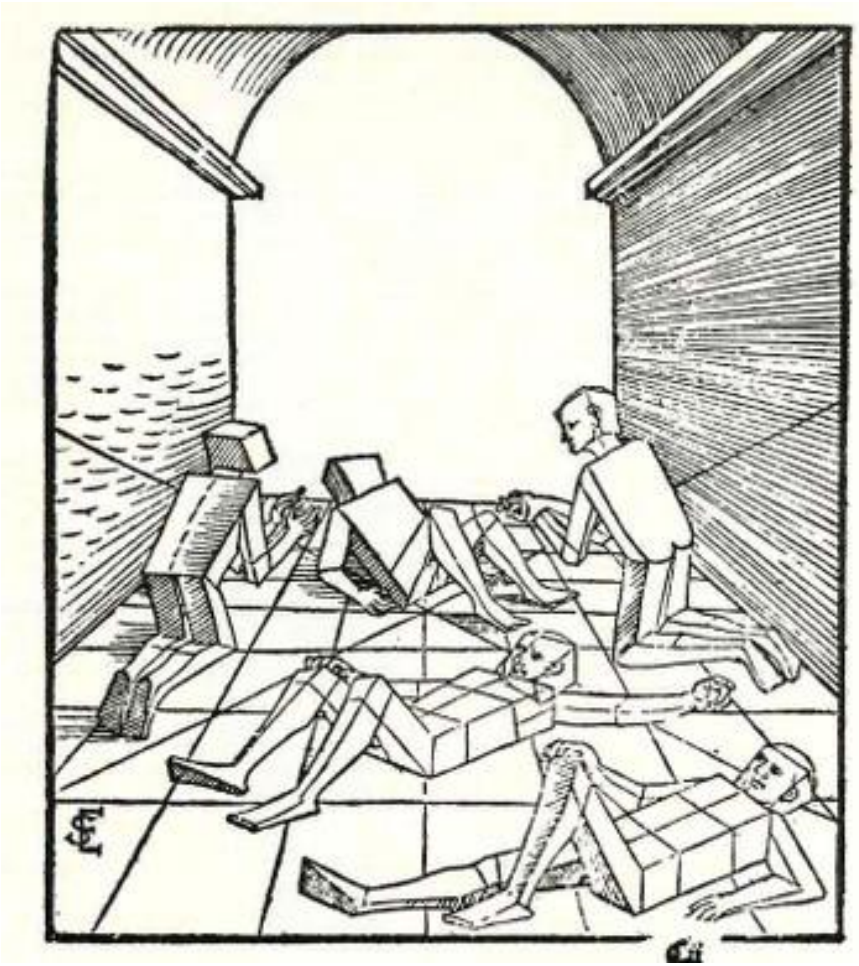
1525

Vnd damit günstiger lieber Herr will ich meinem schreyben end geben / vnd so mir Got genad verleycht die bücher so ich von menschlicher proportion vñ anderen darzů gehörend geschrieben hab mit der zeyt in druck pringen / vnd darynn mengklich gewarnet haben / ob sich yemand vnderstee wurd mir diß außgangen büchlein wider nach zů drucken / das ich das selb auch wider drucken will / vñ auß lassen geen mit meren vnd grösserem zůsatz dañ ien beschehen ist / darnach mag sich ein yetlicher richtē Got dem Herren sey lob vnd ere ewigklich.

57 iij

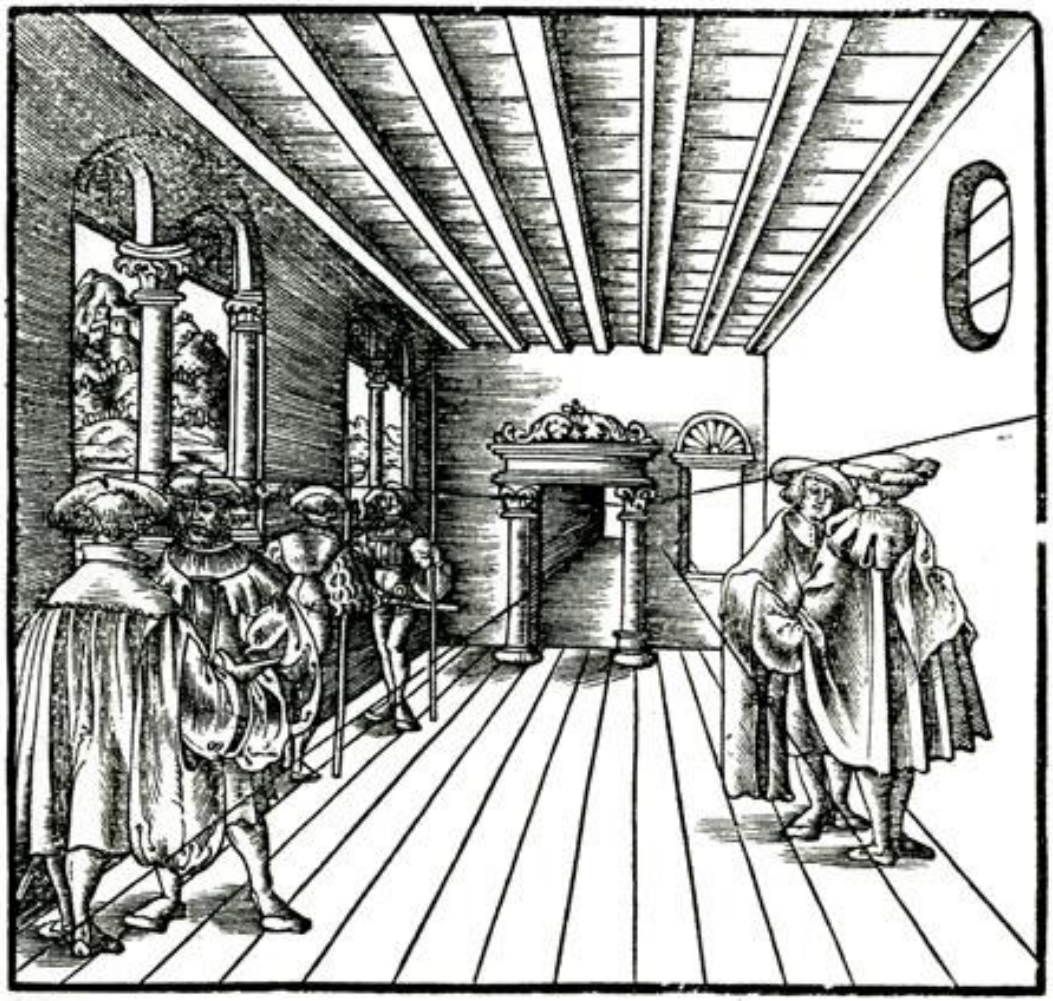
Gedruckt zů Nüremberg.
Im. 1525. Jar.

Erhard Schön (1490-1542) “Placing models on a perspective grid”



Count Johann von Simmern (1492-1557)

Heights of People



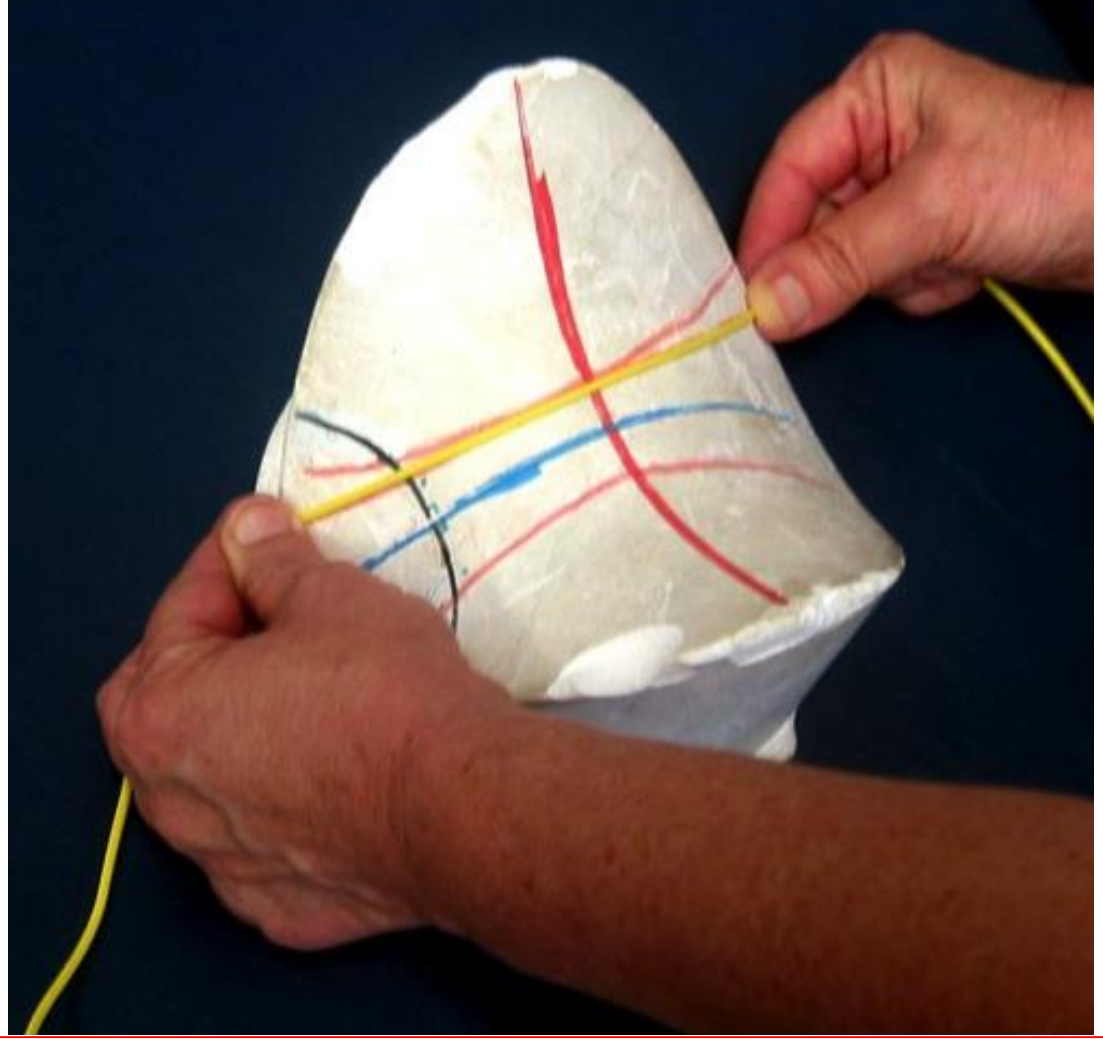
Raphael, The School of Athens, 1509-1510

The three independent discoverers of Non-Euclidean Geometry

Carl Friedrich Gauss (1777 - 1855)

János Bolyai (1802 – 1860)

Nikolai Ivanovich Lobachevsky (1792 – 1856)



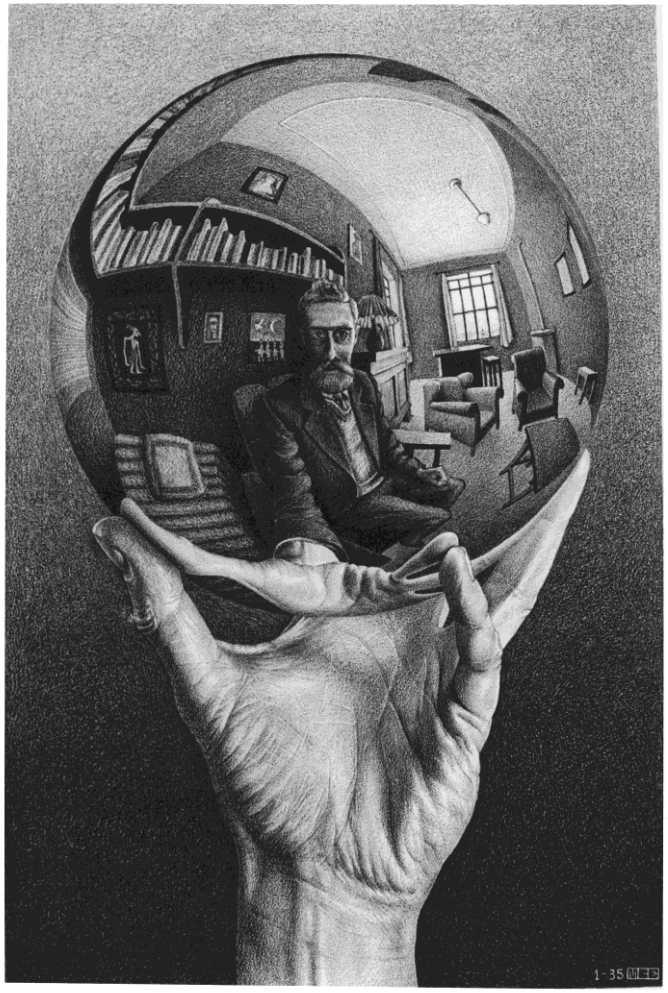
Non-Euclidean Surface: The "**shortest distances**" **red & yellow** never meet the blue, which means both are **parallel to the blue**.
Yellow and blue are parallel to one another (never meet)
but **are not equally distant** from one another.
Both red and yellow, although **parallel to the same blue line,** nevertheless **pass through the same point**.

Georg Bernhard Riemann
(1826 – 1866)

He brought non-Euclidean geometry into the mainstream of mathematics through his theory of more generalized spaces, which he called “manifolds.”

Hermann von Helmholtz, 1821 – 1894

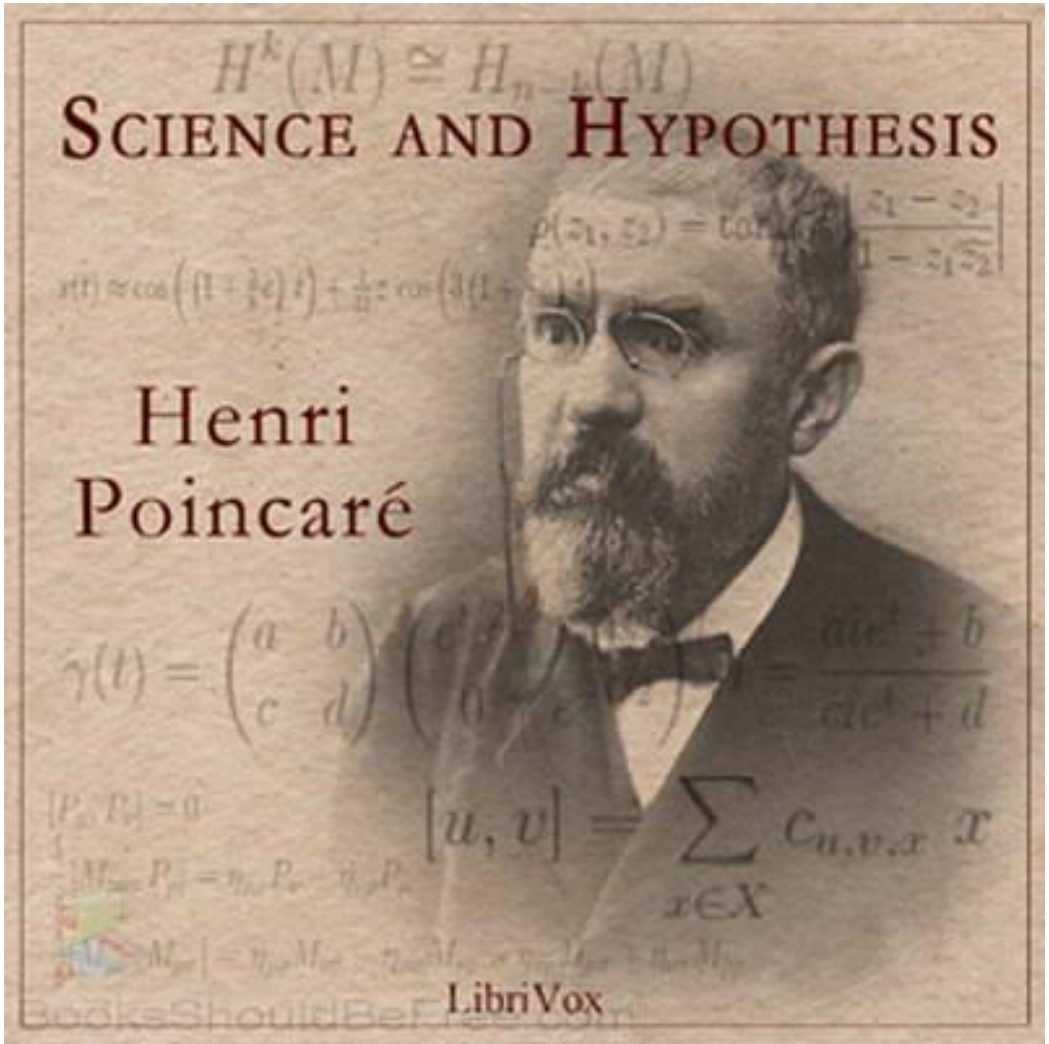
M. C. Escher, “Hand with Reflecting Sphere”



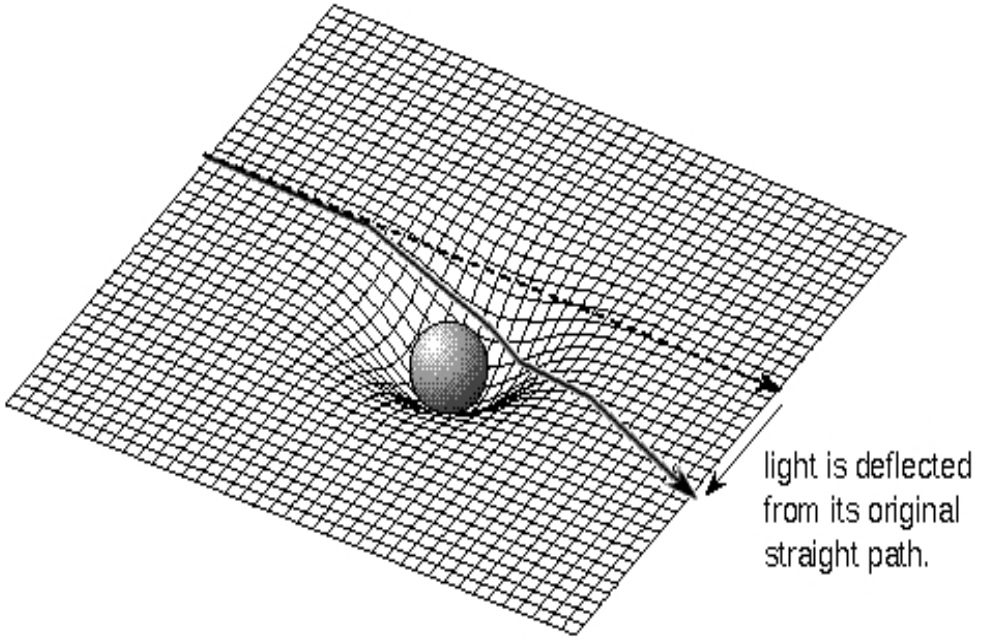
“What Copernicus was to Ptolemy, so Lobachevsky was to Euclid.”

-- W. K. Clifford (1845 – 1879)

Henri Poincaré, 1854 –1912



Einstein, general relativity, curved space



General Relativity: Light travels along the curved space taking the shortest path between two points. Therefore, light is deflected toward a massive object! The stronger the local gravity is, the greater the light path is bent.

Corner of a room

José Ortega y Gasset (1883 – 1955):

Euclidean geometry is **an unwarranted extrapolation of what was locally observed** to the whole universe. Einstein's relativity, which requires new geometries of space-time, promotes the **harmonious multiplicity of all possible points of view**.

Zaha Hadid, “The Ideal House” (2007)

Buildings designed by Zaha Hadid

:

“A non-Euclidean geometry in which nothing repeats itself, a new order of space.”

Jean Metzinger, “Tea Time” (*Le Gouter*) 1911

Juan Gris, “The Guitar” (1918)

Pablo Picasso, “Portrait of Ambroise Vollard”

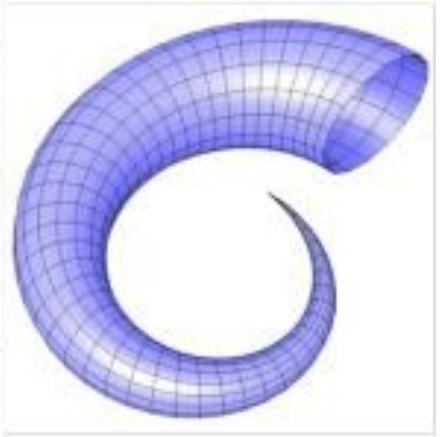
Plaster models of non-Euclidean surfaces, about 1900.

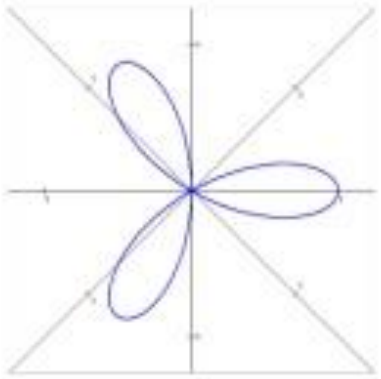
Antoine Pevsner, “Model for ‘World’ ”

Naum Gabo, “Linear Construction #4” (1959)

Man Ray, “Lampshade”

Max Ernst, “Young Man Intrigued by the Flight of a Non-Euclidean Fly” (1942-7)





Wichita, Kansas: Gas Station, 1958

Olympic Velodrome

Hopkins Architects, London, 2012

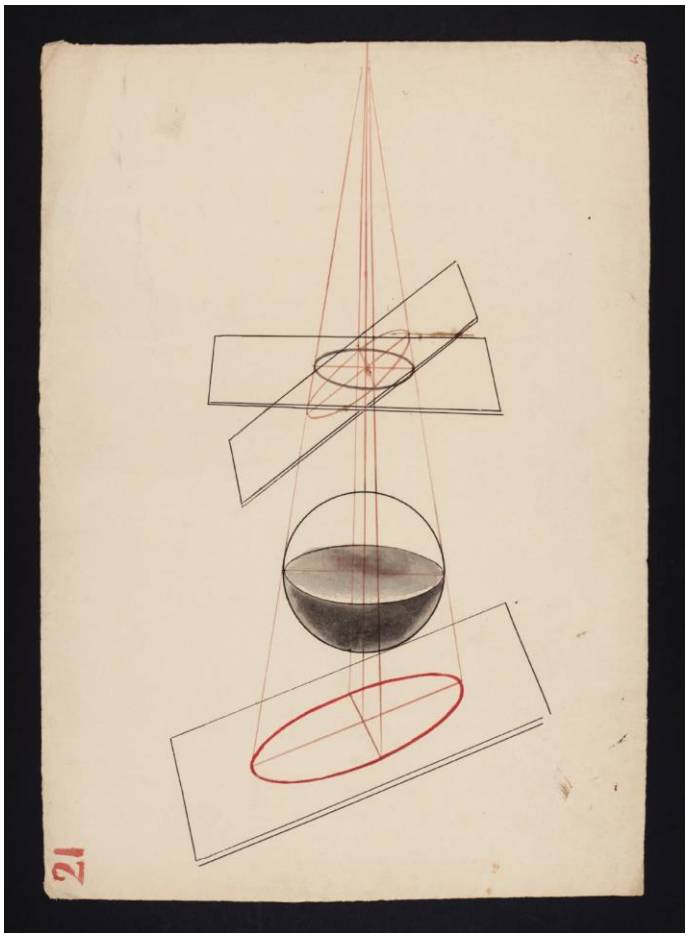
Jan van Eyck, The Arnolfini Wedding, 1434

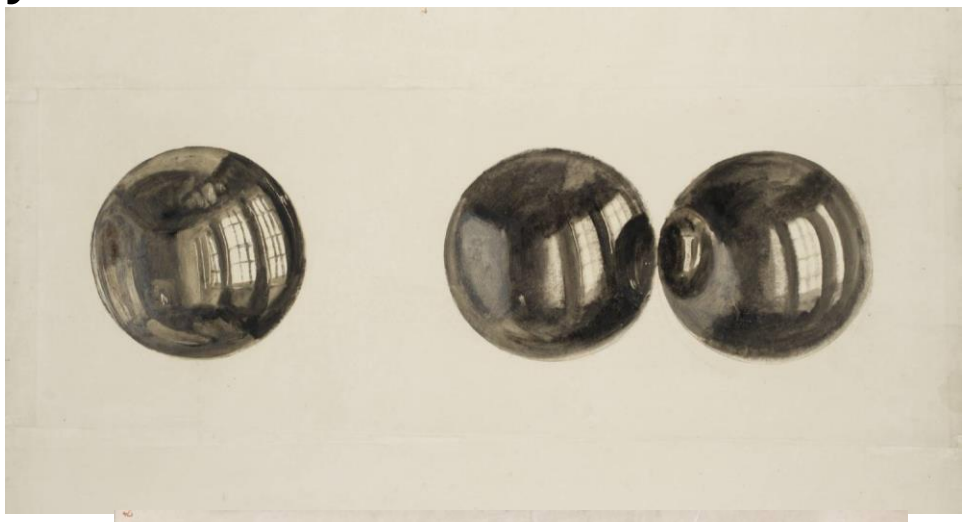
Detail from Arnolfini Wedding

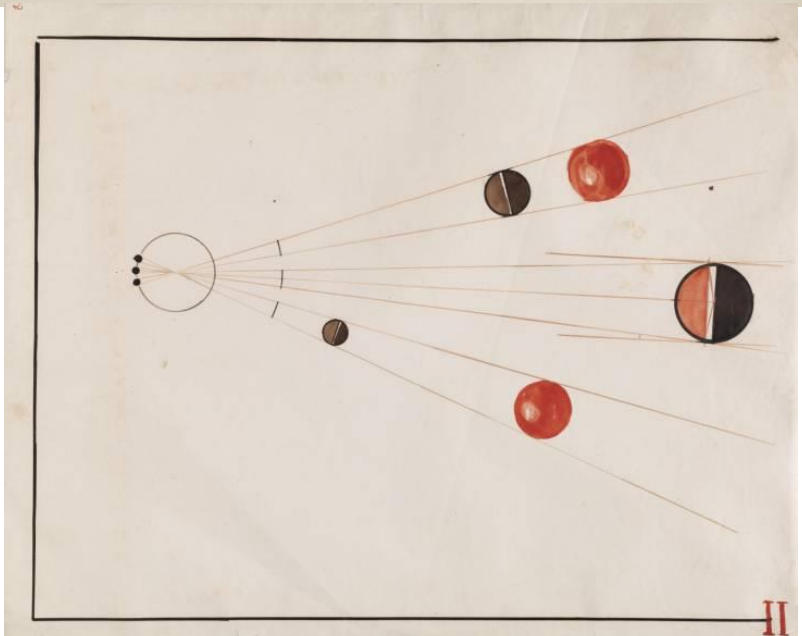
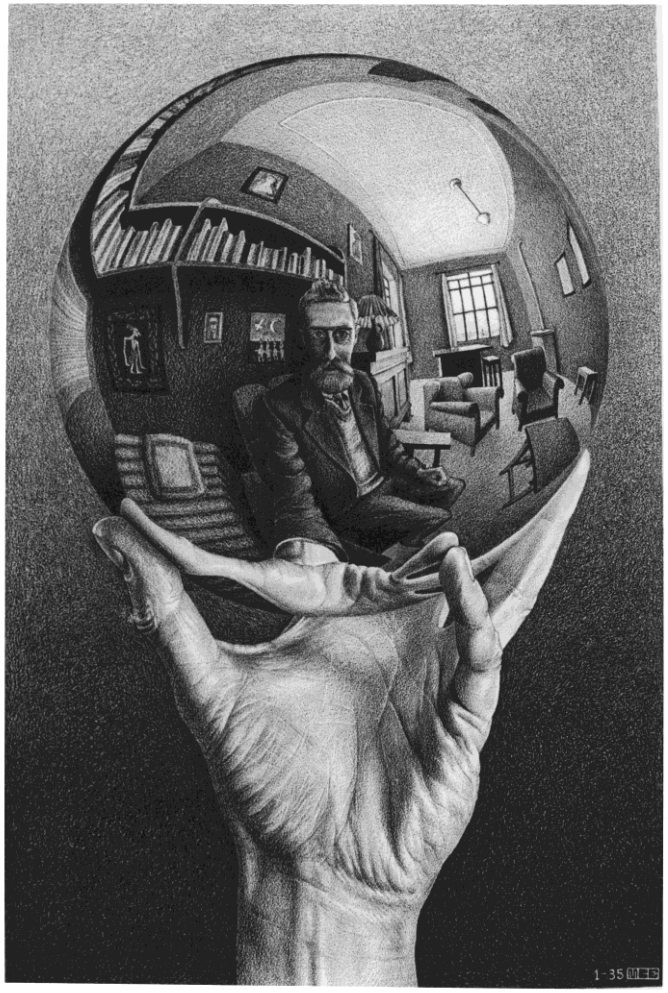
J. M. W. Turner (1775-1851)

Numbered Perspective Diagrams

A globe in perspective, spheres at different distances from the eye, and reflections in metal globes







M. C. Escher, “Hand with Reflecting Sphere”